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A THEORETICAL STUDY OF THE MOMENT ON A BODY IN A COMPRESSIBLE FLUID

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SUMMARY

The extension to a compressible fluid of Lagally's theorem on the moment on a body in an incompressible fluid and Poggi's method of treating the flow of compressible fluids are employed for the determination of the effect of compressibility on the moment on an arbitrary body. Only the case of the two-dimensional subsonic flow of an ideal compressible fluid is considered.

As examples of the application of the general theory, two well-known systems of profiles are treated; namely, the elliptic profile and the symmetrical Joukowski profiles with sharp trailing edges.

The effect of compressibility on the position of the center of pressure is also discussed. In order to determine this effect, it is necessary to calculate the additional circulation induced by the compressibility of the fluid for both the elliptic and the Joukowski profiles. For these two types of profile, the centers of pressure in the compressible and the incompressible fluids are found to coincide for a definite and fairly small angle of attack, which is essentially dependent on the thickness coefficients. For angles of attack less than this neutral angle, the center of pressure in the compressible fluid is farther from the nose and, for angles of attack greater than the neutral angle, nearer to the nose than the center of pressure in the incompressible fluid.

Several numerical examples of both the elliptic and the Joukowski profiles are given. The results show that, although the effect of compressibility on the moment and on the lift may be large, the effect on the center of pressure for conventional profiles is negligible. Thus, for a Joukowski profile, the maximum thickness of which is equal to 18 percent of the chord, the center of pressure moves toward the nose a distance equal to only 0.19 percent of the chord, where the angle of attack is 6° and $v_0/c_0=0.70$.

NOTATION

GENERAL SYMBOLS

ξ, η, rectangular Cartesian coordinates in the plane of the obstacle.

x, y, rectangular Cartesian coordinates in the plane of the circle.

$\zeta=\xi+i\eta,\ z=x+iy$

r, θ, polar coordinates in the plane of the circle.

R, radius vector of a point far removed from the obstacle and also of a point far removed from the corresponding circle.

r_0, radius of circle into which the profile is mapped.

$\lambda=\frac{r_0}{r}$

v_r, v_θ, components of the velocity in the radial and the circumferential directions in the z plane.

v, magnitude of the velocity in the plane of the obstacle.

c, magnitude of the local velocity of sound.

ρ, density of the fluid.

p, static pressure of the fluid.

v_0, c_0, ρ_0, p_0, corresponding magnitudes in the undisturbed part of the fluid.

γ, ratio of the specific heats (c_p/c_v).

$\mu=\left(\frac{v_0}{c_0}\right)^2$, square of the Mach number.

β, angle of attack.

Γ, circulation about the obstacle.

$K=\frac{\Gamma}{2\pi r_0 v_0}$

$L=\rho_0 v_0 \Gamma$, lifting force on the obstacle.

M, moment on the obstacle.

c_c, c_i, centers of pressure, respectively, in the compressible and the incompressible fluids.

Subscripts c and i refer, respectively, to the compressible and the incompressible fluids.

$\Delta v_R, \Delta v_\delta$, additional velocity components due to compressibility at a point $P(R, \delta)$ far removed from the circle in the z plane.

w, complex velocity potential of the fluid.

$\frac{dw}{d\zeta}$, complex velocity in the plane of the obstacle.

$A_n, \bar{A}_n$, complex and conjugate complex coefficients, respectively, in the power series development of $\frac{dw}{d\zeta}$ (see equation (12)).

SYMBOLS PERTAINING TO ELLIPTIC PROFILE

c, semifocal distance.

t, thickness ratio of ellipse (ratio of semiminor and semimajor axes).

$\sigma^2=\frac{1-t}{1+t}$, where $\sigma=\frac{c}{2r_0}$.

$\tanh \alpha=t$

M, N, functions of the thickness coefficient t only (see equation (24)).

SYMBOLS PERTAINING TO JOUKOWSKI PROFILES

ϵ, thickness coefficient (see fig. 4).

$h=\frac{\epsilon}{1+\epsilon}$

$k=\frac{1-\epsilon}{1+\epsilon}=1-2h$

I, J, (see equation (26)).

M, N, functions of the thickness coefficient ϵ only (see equation (29)).

DERIVATION OF THE FORMULA FOR THE MOMENT

Theodorsen's method (reference 1) of extending Lagally's formula for the force on a body in an incompressible fluid to a compressible fluid may be used to obtain the corresponding formula for the moment. The body is fixed in an infinite two-dimensional stream of a frictionless compressible fluid flowing uniformly in the far field. Then the moment exerted on the body with respect to the origin of the coordinate system is given by (fig. 1)

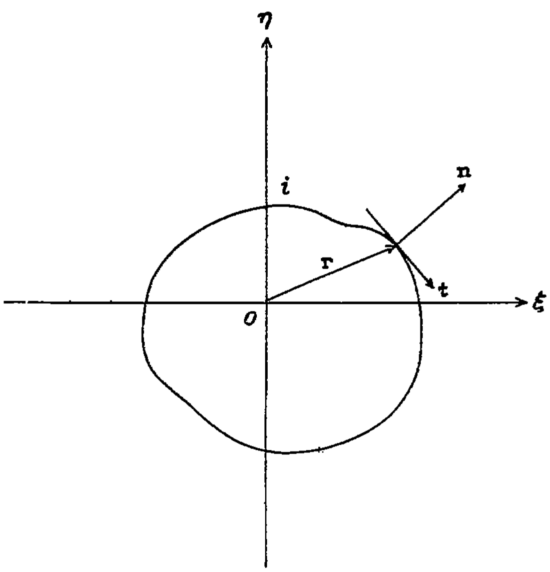

FIGURE 1.—Directions of tangent and normal on a contour.

$$M=-\int_i [\mathbf{r}\ \mathbf{n}]p\ ds \qquad (1)$$

where the positive direction of the unit normal vector $\mathbf{n}$ is taken from the boundary into the fluid, p is the pressure of the fluid, $[\mathbf{r}\ \mathbf{n}]$ is the vector product of the radius vector $\mathbf{r}$ and the unit normal vector $\mathbf{n}$, and ds is the element of length along the profile taken positively in the direction of the unit tangent vector $\mathbf{t}$. With the directions of the unit tangent vector $\mathbf{t}$ and the unit normal vector $\mathbf{n}$ so chosen that they form a right-hand system, a positive value for the moment corresponds to a counterclockwise rotation.

According to one of the generalizations from Gauss' theorem, it may be shown that

$$\int_s \text{curl}\ p\mathbf{r}dS=\int_i[\mathbf{r}\ \mathbf{n}]p\ ds+\int_o[\mathbf{r}\ \mathbf{n}]p\ ds$$

where dS is the element of surface in a region S included between the obstacle i and an arbitrary curve o enclosing it.

Now

$$\text{curl}\ p\mathbf{r}=[\text{grad}\ p\ \mathbf{r}]$$

and, from the Euler equations of motion for steady flow (reference 1),

$$\text{grad}\ p=-1/2\ \rho\ \text{grad}\ v^2$$

Therefore

$$\text{curl}\ p\mathbf{r}=-1/2\ \rho\ [\text{grad}\ v^2\ \mathbf{r}]$$

Equation (1) then becomes

$$M=\int_o [\mathbf{r}\ \mathbf{n}]\ p\ ds+1/2\int_s \rho\ [\text{grad}\ v^2\ \mathbf{r}]\ dS$$

Since the outer boundary is arbitrary, it may be chosen to be a large circle so that the vector product $[\mathbf{r}\ \mathbf{n}]=0$. Hence

$$M=1/2\int_s \rho\ [\text{grad}\ v^2\ \mathbf{r}]\ dS$$

But

$$\text{grad}\ v^2=2\ (\mathbf{v}\ \text{grad})\ \mathbf{v}+2\ [\mathbf{v}\ \text{curl}\ \mathbf{v}]$$

and

$$[(\mathbf{v}\ \text{grad})\ \mathbf{v}\ \mathbf{r}]=(\mathbf{v}\ \text{grad})\ [\mathbf{v}\ \mathbf{r}]$$

Therefore

$$M=\int_s \rho(\mathbf{v}\ \text{grad})[\mathbf{v}\ \mathbf{r}]dS+\int_s \rho\{[\mathbf{v}\ \text{curl}\ \mathbf{v}]\mathbf{r}\}dS \qquad (2)$$

According to Gauss' theorem, if F denotes a scalar function, then

$$\int_s \text{div}\ \rho F\mathbf{v}dS=-\int_i \rho F(\mathbf{v}\ \mathbf{n})ds-\int_o \rho F(\mathbf{v}\ \mathbf{n})ds$$

But

$$\text{div}\ \rho F\mathbf{v}=F\ \text{div}\ \rho\mathbf{v}+\rho(\mathbf{v}\ \text{grad})F$$

Hence

$$\int_s \rho(\mathbf{v}\ \text{grad})FdS=-\int_s F\ \text{div}\ \rho\mathbf{v}dS -\int_i \rho F(\mathbf{v}\ \mathbf{n})ds-\int_o \rho F(\mathbf{v}\ \mathbf{n})ds$$

From the manner in which F occurs in this formula, it is clear that the formula remains valid if F is replaced by a vector point function, say $[\mathbf{v}\ \mathbf{r}]$. That is,

$$\int_s \rho(\mathbf{v}\ \text{grad})[\mathbf{v}\ \mathbf{r}]dS=\int_s [\mathbf{r}\ \mathbf{v}]\ \text{div}\ \rho\mathbf{v}dS +\int_i \rho[\mathbf{r}\ \mathbf{v}](\mathbf{v}\ \mathbf{n})ds+\int_o \rho[\mathbf{r}\ \mathbf{v}](\mathbf{v}\ \mathbf{n})ds$$

Substituting from this equation into equation (2), it follows that

$$M=\int_o \rho[\mathbf{r}\ \mathbf{v}](\mathbf{v}\ \mathbf{n})ds+\int_s [\mathbf{r}\ \mathbf{v}]\ \text{div}\ \rho\mathbf{v}dS -\int_s \rho\{\mathbf{r}[\mathbf{v}\ \text{curl}\ \mathbf{v}]\}dS \qquad (3)$$

where $\int_i \rho[\mathbf{r}\ \mathbf{v}](\mathbf{v}\ \mathbf{n})ds=0$ since $(\mathbf{v}\ \mathbf{n})=0$ everywhere on the obstacle.[1]

In the problem considered herein, the compressible fluid is of uniform velocity v_0 in the undisturbed stream

[1] This formula for the moment may be extended to three dimensions by letting τ denote the region of flow between the body and a large sphere enclosing it; ds is then replaced by $d\sigma$, the element of surface on the outer sphere, and dS is replaced by $d\tau$, the element of volume in the region of flow τ. Also, the moment now has three components and must be written as a vector, i. e., $\mathbf{M}$.

and, furthermore, is assumed to be irrotational and free of sources in the region of flow S. The condition for irrotational motion is simply that curl $\mathbf{v}=0$, and the absence of sources means that div $\rho\,\mathbf{v}=0$. Therefore, equation (3) becomes

$$M=\int_0 \rho\,[\mathbf{r}\ \mathbf{v}]\ (\mathbf{v}\ \mathbf{n})\ ds \tag{4}$$

where it is recalled that the outer boundary has been assumed to be a large circle of radius R. The vector product $[\mathbf{r}\ \mathbf{v}]$ is then equal to Rv_φ and the scalar product $(\mathbf{v}\ \mathbf{n})$ is equal to $-v_R$ since $\mathbf{n}=-\mathbf{R}/R$. Therefore

$$M=-R^2\int_0^{2\pi}\rho v_R v_\varphi d\varphi \tag{5}$$

This simple expression for the moment about the origin, of the resultant force acting on the body, can be directly obtained by considering the rate at which angular momentum passes out of the region included between the outer circle and the body. Thus, the momentum per unit time passing normally through the element ds is $\rho\,Rv\,v_R\,d\varphi$ and the arm is R sin $\mathbf{vR}$. The angular momentum is therefore $\rho R^2\,v$ sin $\mathbf{vR}\ v_R\ d\varphi$ or $\rho R^2\,v_R\,v_\varphi\,d\varphi$ and equation (5) follows.

GENERAL DEVELOPMENTS

It may be well to emphasize at this point that the main problem of this paper lies in obtaining useful expansions for the velocity components v_R and v_φ of a compressible fluid.

These expansions are obtained by making use of Poggi's conception of compressible flow. Thus the basic differential equation for the steady flow of a compressible fluid may be written as (reference 2)

$$\frac{\partial v_\xi}{\partial \xi}+\frac{\partial v_\eta}{\partial \eta}=\frac{1}{2c^2}\left(v_\xi\frac{\partial v^2}{\partial \xi}+v_\eta\frac{\partial v^2}{\partial \eta}\right)$$

The expression on the left-hand side is div $\mathbf{v}$, so that the expression on the right-hand side may be considered to represent a source distribution of a strength given by

$$\frac{1}{4\pi c^2}\left(v_\xi\frac{\partial v^2}{\partial \xi}+v_\eta\frac{\partial v^2}{\partial \eta}\right)d\xi\ d\eta$$

In the plane of the circle into which the profile of the obstacle is mapped by a suitable conformal transformation, the strength of the source distribution may be written as

$$\frac{1}{4\pi c^2}\left(v_r\frac{\partial v^2}{\partial \lambda}-\frac{v_\theta}{\lambda}\frac{\partial v^2}{\partial \theta}\right)\frac{r_0}{\lambda}d\lambda\ d\theta \tag{6}$$

where

- r, θ are the polar coordinates of a point in the plane $z(=x+iy)$ of the circle.
- $\lambda=\frac{r_0}{r}$, where r_0 is the radius of the circle into which the profile is mapped.
- $v_r=-\frac{\partial \phi}{\partial r}$, $v_\theta=-\frac{1}{r}\frac{\partial \phi}{\partial \theta}$, where ϕ is the velocity potential of the flow.
- v, the magnitude of the velocity of the fluid in the plane of the profile.
- c, the magnitude of the local velocity of sound.

Poggi's method of approximating the flow of a perfect compressible fluid past an obstacle is based on the fact that the incompressible fluid is a good first approximation. A first approximation for the source distribution given by equation (6) is therefore obtained by substituting for v_r, v_θ, and v^2 the values belonging to the incompressible fluid. It is further assumed that the Mach number v_0/c_0 is small so that only terms involving the lowest order of v_0/c_0 are to be considered. This assumption then limits the application of the analysis to stream velocities small in comparison with the velocity of sound. The disturbances to the main flow due to the presence of a body in the fluid may, however, be large.[2]

The density ρ and the pressure p of the fluid are then determined by the following equations (reference 2):

$$\left.\begin{aligned}\rho&=\rho_0\left[1+\frac{\gamma-1}{2}\mu\left(1-\frac{v^2}{v_0^2}\right)\right]^{\frac{1}{\gamma-1}}\\&=\rho_0\left[1+1/2\mu\left(1-\frac{v_i^2}{v_0^2}\right)+\ldots\right]\\p&=\text{constant}-\frac{1}{2}\rho_0 v_c^2-\frac{1}{4}\mu\rho_0 v_i^2+\frac{1}{8}\mu\frac{\rho_0 v_i^4}{v_0^2}+\ldots\end{aligned}\right\} \tag{7}$$

where the adiabatic equation of state has been adopted. Also, $\mu=(v_0/c_0)^2$ and v_i and v_c are, respectively, the velocities in the incompressible and the compressible fluids.

It follows from equation (5), with the stipulation that only terms involving the square of the Mach number are to be retained, that:

$$\begin{aligned}M_c=&-\rho_0 v_0^2R^2\int_0^{2\pi}\left(\frac{v_R}{v_0}\frac{v_\varphi}{v_0}\right)_c d\varphi\\&-\frac{1}{2}\mu\rho_0v_0^2R^2\int_0^{2\pi}\left(1-\frac{v_i^2}{v_0^2}\right)\left(\frac{v_R}{v_0}\frac{v_\varphi}{v_0}\right)_i d\varphi\end{aligned} \tag{8}$$

It is clear that the evaluation of the second integral presents no difficulties, for it involves only expressions of the well-known incompressible fluid-velocity components. The difficulty of the problem lies mainly in determining the velocity components v_R and v_φ for a compressible fluid. Since the first integral is taken around a circle whose radius R may be infinite, it is obvious that the developments for v_R/v_0 and v_φ/v_0 in the neighborhood of infinity need not go beyond the $1/R^2$ terms.

In order to obtain the series for v_R/v_0 and v_φ/v_0, it is expedient to consider first the effect of a single source of unit strength situated at a point $Q(r, \theta)$ in the plane of a circle. In the presence of a circular boundary of radius r_0, the velocity induced by a unit source at any

[2] In the approximation made by Glauert (reference 3), v_0 is not small compared with c_0 but the disturbances to the main flow v_0 are assumed to be small.

point $P(R, \delta)$ external to the boundary is given by (fig. 2):

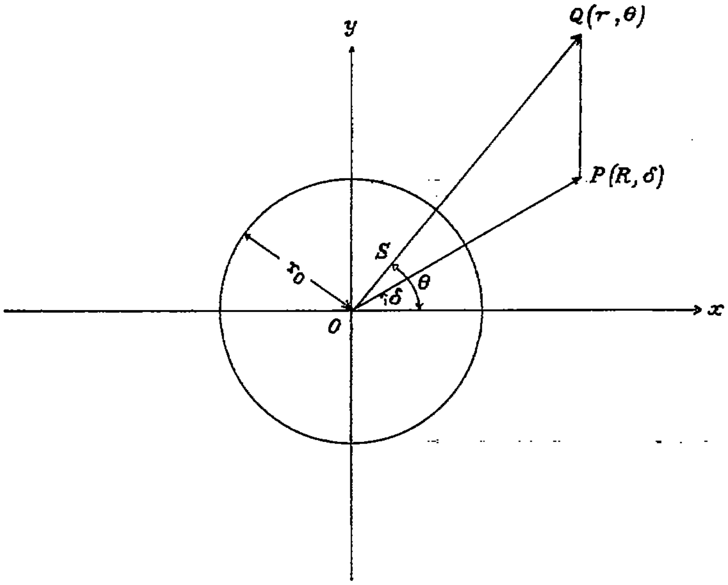

FIGURE 2.—Image of a simple source with regard to a circle.

$$\frac{dw}{dz}=-\left(\frac{1}{z_P-z_Q}+\frac{1}{z_P-z_S}-\frac{1}{z_P}\right)=-v_x+iv_y$$

where S is the point inverse to Q with respect to the circle;

$$z_P=Re^{i\delta};\ z_Q=re^{i\theta};\ \text{and}\ z_S=\frac{r_0^2}{r}e^{i\theta}.$$

Then

$$v_x-iv_y=\frac{1}{Re^{i\delta}-re^{i\theta}}+\frac{1}{Re^{i\delta}-\frac{r_0^2}{r}e^{i\theta}}-\frac{1}{Re^{i\delta}}$$

or since

$$(-v_R+iv_\delta)e^{-i\delta}=-v_x+iv_y$$

for $r_0<r<R$

$$v_R=\frac{1}{2R}\frac{1-\frac{\lambda_P^2}{\lambda^2}}{1-2\frac{\lambda_P}{\lambda}\cos(\theta-\delta)+\frac{\lambda_P^2}{\lambda^2}}$$

$$+\frac{1}{2R}\frac{1-\lambda_P^2\lambda^2}{1-2\lambda_P\lambda\cos(\theta-\delta)+\lambda_P^2\lambda^2}$$

$$v_\delta=-\frac{1}{R}\frac{\frac{\lambda_P}{\lambda}\sin(\theta-\delta)}{1-2\frac{\lambda_P}{\lambda}\cos(\theta-\delta)+\frac{\lambda_P^2}{\lambda^2}}$$

$$-\frac{1}{R}\frac{\lambda_P\lambda\sin(\theta-\delta)}{1-2\lambda_P\lambda\cos(\theta-\delta)+\lambda_P^2\lambda^2}$$

and for $R<r<\infty$

$$v_R=-\frac{1}{2R}\frac{1-\frac{\lambda^2}{\lambda_P^2}}{1-2\frac{\lambda}{\lambda_P}\cos(\theta-\delta)+\frac{\lambda^2}{\lambda_P^2}}$$

$$+\frac{1}{2R}\frac{1-\lambda_P^2\lambda^2}{1-2\lambda_P\lambda\cos(\theta-\delta)+\lambda_P^2\lambda^2}$$

$$v_\delta=-\frac{1}{R}\frac{\frac{\lambda}{\lambda_P}\sin(\theta-\delta)}{1-2\frac{\lambda}{\lambda_P}\cos(\theta-\delta)+\frac{\lambda^2}{\lambda_P^2}}$$

$$-\frac{1}{R}\frac{\lambda_P\lambda\sin(\theta-\delta)}{1-2\lambda_P\lambda\cos(\theta-\delta)+\lambda_P^2\lambda^2}$$

where $\lambda=r_0/r$ and $\lambda_P=r_0/R$.

Then, making use of the expansion

$$\frac{1-q^2}{1-2q\cos(\theta-\delta)+q^2}=1+2\sum_{n=1}^{\infty}q^n\cos n(\theta-\delta)$$

it follows that the components of the velocity induced at any point $P(R, \delta)$ by the source distribution given by equation (6) are:

$$\frac{\Delta v_R}{v_0}=-\frac{\mu}{4\pi}\int_{\lambda_P}^{1}\int_0^{2\pi}F\frac{\lambda_P}{\lambda}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{\lambda_P}{\lambda}\right)^n\cos n(\theta-\delta)\right]d\lambda\,d\theta$$

$$+\frac{\mu}{4\pi}\int_0^{\lambda_P}\int_0^{2\pi}F\frac{\lambda_P}{\lambda}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{\lambda}{\lambda_P}\right)^n\cos n(\theta-\delta)\right]d\lambda\,d\theta$$

$$-\frac{\mu}{4\pi}\int_0^{1}\int_0^{2\pi}F\frac{\lambda_P}{\lambda}\left[\frac{1}{2}+\sum_{n=1}^{\infty}(\lambda_P\lambda)^n\cos n(\theta-\delta)\right]d\lambda\,d\theta \quad (9)$$

and

$$\frac{\Delta v_\delta}{v_0}=\frac{\mu}{4\pi}\int_{\lambda_P}^{1}\int_0^{2\pi}F\frac{\frac{\lambda_P^2}{\lambda^2}\sin(\theta-\delta)}{1-2\frac{\lambda_P}{\lambda}\cos(\theta-\delta)+\frac{\lambda_P^2}{\lambda^2}}d\lambda\,d\theta$$

$$+\frac{\mu}{4\pi}\int_0^{\lambda_P}\int_0^{2\pi}F\frac{\sin(\theta-\delta)}{1-2\frac{\lambda}{\lambda_P}\cos(\theta-\delta)+\frac{\lambda^2}{\lambda_P^2}}d\lambda\,d\theta$$

$$+\frac{\mu}{4\pi}\int_0^{1}\int_0^{2\pi}F\frac{\lambda_P^2\sin(\theta-\delta)}{1-2\lambda_P\lambda\cos(\theta-\delta)+\lambda_P^2\lambda^2}d\lambda\,d\theta \quad (10)$$

where

$$F=\left(\frac{v_r}{v_0}\frac{\partial\frac{v^2}{v_0^2}}{\partial\lambda}-\frac{v_\theta}{v_0}\frac{1}{\lambda}\frac{\partial\frac{v^2}{v_0^2}}{\partial\theta}\right)_i$$

The next step is to obtain the Fourier series expansion of the function F for an arbitrary profile. The profile is derived from a circle by means of the general conformal transformation

$$\zeta=z'+\frac{a_1}{z'}+\frac{a_2}{z'^2}+\ldots$$

where the coefficients a_i are, in general, complex. In order to obtain the velocity of the fluid in the plane of the profile, it is first necessary to determine the velocity potential of the flow past a circle (fig. 3). Thus, suppose that the undisturbed flow is of velocity v_0 inclined at an angle β to the negative direction of the real axis and that the circulation is denoted by Γ. In terms of

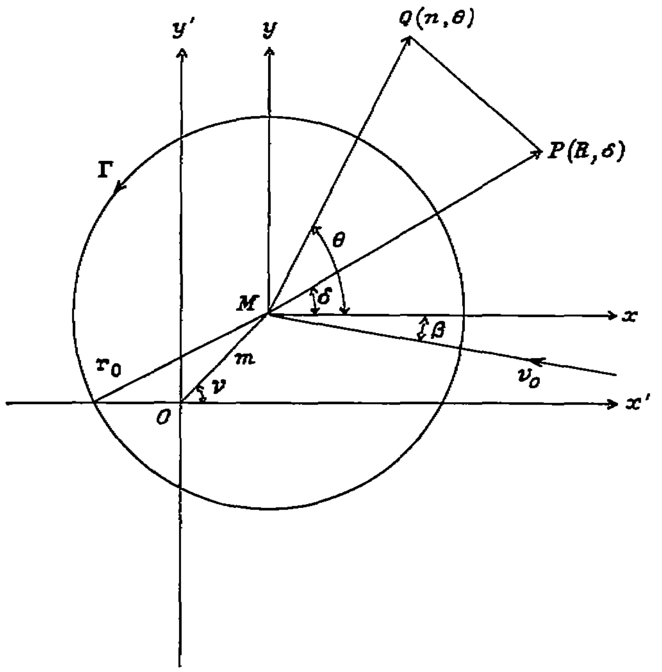

FIGURE 3.—Relation between z planes employed in general conformal transformation.

the complex coordinate z, with origin at the center of the circle, the potential function of the flow past the circle is:

$$w=v_0\left(ze^{i\beta}+\frac{r_0^2}{ze^{i\beta}}\right)+\frac{i\Gamma}{2\pi}\log\frac{ze^{i\beta}}{r_0} \qquad (11)$$

Also, the variables z and z' are connected by the equation

$$z'=z+me^{i\nu}$$

Therefore

$$\frac{dw}{d\zeta}=\frac{dw}{dz}\frac{dz}{dz'}\frac{dz'}{d\zeta}=\frac{v_0e^{i\beta}-\dfrac{v_0r_0^2e^{-i\beta}}{z^2}+\dfrac{i\Gamma}{2\pi z}}{1-\dfrac{a_1}{z'^2}-\dfrac{2a_2}{z'^3}-\ \ldots}$$

Substituting for z' in terms of z and expanding in descending powers of z, this expression becomes:

$$\frac{1}{v_0}\frac{dw}{d\zeta}=e^{i\beta}+\frac{i\Gamma}{2\pi v_0}\frac{1}{z}+(a_1e^{i\beta}-r_0^2e^{-i\beta})\frac{1}{z^2}+\ \ldots \qquad (12)$$

It follows then, that:

$$\frac{1}{v_0}\frac{dw}{d\zeta}=\sum_{n=0}^{\infty}\frac{A_n}{z^n}$$

where the A_n's are, in general, complex.

In order to introduce the angle $\theta-\delta$, the general term of this series is multiplied by $e^{-in\delta}/e^{-in\delta}$. Then

$$\frac{1}{v_0}\frac{dw}{d\zeta}=\sum_{k=0}^{\infty}\frac{A_ke^{-ik\delta}}{r^ke^{ik(\theta-\delta)}}$$

and

$$\frac{1}{v_0}\overline{\frac{dw}{d\zeta}}=\sum_{j=0}^{\infty}\frac{\overline{A}_je^{ij\delta}}{r^je^{-ij(\theta-\delta)}}$$

Hence

$$\frac{v^2}{v_0^2}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{A_k\overline{A}_je^{i(j-k)\delta}}{r^{j+k}}e^{i(j-k)(\theta-\delta)}$$

The real and the imaginary parts of $A_k\overline{A}_je^{i(j-k)\delta}$ are, respectively:

$$\frac{A_k\overline{A}_je^{i(j-k)\delta}+\overline{A}_kA_je^{-i(j-k)\delta}}{2}$$

and

$$-i\frac{A_k\overline{A}_je^{i(j-k)\delta}-\overline{A}_kA_je^{-i(j-k)\delta}}{2}$$

Let

$$j-k=n$$

where n changes sign when j and k are interchanged and therefore takes values from $-\infty$ to ∞.
Then

$$j+k=n+2k$$

and the expression for v^2/v_0^2, a real quantity, becomes:

$$\frac{v^2}{v_0^2}=\sum_{k=0}^{\infty}\sum_{n=-\infty}^{\infty}[A_{n,k}\cos n(\theta-\delta)+B_{n,k}\sin n(\theta-\delta)]\lambda^{n+2k} \quad (13)$$

where

$$A_{n,k}=\frac{A_k\overline{A}_{n+k}e^{in\delta}+\overline{A}_kA_{n+k}e^{-in\delta}}{2r_0^{\ n+2k}}$$

and

$$B_{n,k}=i\frac{A_k\overline{A}_{n+k}e^{in\delta}-\overline{A}_kA_{n+k}e^{-in\delta}}{2r_0^{\ n+2k}}$$

From the definition of n, it follows that:

$$A_{-n,k}=A_{n,k}$$

and

$$B_{-n,k}=-B_{n,k}$$

The terms of equation (13) can therefore be grouped in pairs, thus:

$$\frac{v^2}{v_0^2}=\sum_{k=0}^{\infty}\lambda^{2k}A_{0,k}+2\sum_{n=1}^{\infty}\cos n(\theta-\delta)\sum_{k=0}^{\infty}A_{n,k}\lambda^{n+2k}$$
$$+2\sum_{n=1}^{\infty}\sin n(\theta-\delta)\sum_{k=0}^{\infty}B_{n,k}\lambda^{n+2k} \qquad (14)$$

Also, from equation (11),

$$\left.\begin{aligned}\frac{v_r}{v_0}&=-(1-\lambda^2)\cos(\theta-\delta)\cos(\delta+\beta)\\&\quad+(1-\lambda^2)\sin(\theta-\delta)\sin(\delta+\beta)\\ &\text{and}\\ \frac{v_\theta}{v_0}&=(1+\lambda^2)\cos(\theta-\delta)\sin(\delta+\beta)\\&\quad+(1+\lambda^2)\sin(\theta-\delta)\cos(\delta+\beta)+\lambda K\end{aligned}\right\} \qquad (15)$$

where

$$K=\frac{\Gamma}{2\pi r_0v_0}$$

Then, by means of equations (14) and (15), it becomes a simple matter to obtain the Fourier series for the function F. When the Fourier series for F is substituted into equations (9) and (10) and all terms containing powers of λ_P higher than the second are neglected, it follows, after some rather tedious but elementary integrations, that:

$$\begin{aligned}\frac{\Delta v_R}{v_0}=&-\frac{\mu}{2}\lambda_P[A_{1,0}\cos(\delta+\beta)-B_{1,0}\sin(\delta+\beta)]\\&-\frac{\mu}{2}[A_{0,1}\cos(\delta+\beta)+KB_{1,0}]\lambda_P^2\log\lambda_P\\&-\mu\lambda_P^2\Big[\frac{3}{8}A_{0,1}\cos(\delta+\beta)+\frac{1}{2}A_{2,0}\cos(\delta+\beta)\\&-B_{2,0}\sin(\delta+\beta)\Big]+\frac{\mu K}{4}\lambda_P^2\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}B_{1,n}\\&+\frac{\mu}{2}\lambda_P^2\cos(\delta+\beta)\sum_{n=1}^{\infty}\frac{n+1}{n(n+2)}A_{0,n+1}\\&-\frac{\mu}{2}\lambda_P^2\sum_{n=1}^{\infty}\frac{A_{2,n}\cos(\delta+\beta)-B_{2,n}\sin(\delta+\beta)}{n+1}\end{aligned}\tag{16}$$

and

$$\begin{aligned}\frac{\Delta v_\delta}{v_0}=&\frac{\mu}{2}\lambda_P[A_{1,0}\sin(\delta+\beta)-B_{1,0}\cos(\delta+\beta)]\\&-\frac{\mu}{2}[A_{0,1}\sin(\delta+\beta)+KA_{1,0}]\lambda_P^2\log\lambda_P\\&+\mu\lambda_P^2\Big[\frac{1}{8}A_{0,1}\sin(\delta+\beta)+\frac{3}{2}A_{2,0}\sin(\delta+\beta)+\frac{1}{2}KA_{1,0}\Big]\\&+\frac{\mu K}{4}\lambda_P^2\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}A_{1,n}\\&+\frac{\mu}{2}\lambda_P^2\sin(\delta+\beta)\sum_{n=1}^{\infty}\frac{n+1}{n(n+2)}A_{0,n+1}\\&+\frac{\mu}{2}\lambda_P^2\sum_{n=1}^{\infty}\frac{A_{2,n}\sin(\delta+\beta)+B_{2,n}\cos(\delta+\beta)}{n+1}\end{aligned}\tag{17}$$

Use has been made of the following definite integrals in obtaining $\Delta v_\delta/v_0$:

$$\int_0^{2\pi}\frac{\sin(\theta-\delta)\sin n(\theta-\delta)d\theta}{1-2h\cos(\theta-\delta)+h^2}=\begin{bmatrix}0 & \text{if } n=0\\ \pi h^{n-1} & \text{if } n\geqq 1\end{bmatrix}$$

$$\int_0^{2\pi}\frac{\sin(\theta-\delta)\cos n(\theta-\delta)d\theta}{1-2h\cos(\theta-\delta)+h^2}=0$$

The velocity of the compressible fluid in the plane of the circle then has the following components:

$$\left.\begin{aligned}\left(\frac{v_R}{v_0}\right)_c&=-(1-\lambda_P^2)\cos(\delta+\beta)+\frac{\Delta v_R}{v_0}\\ \text{and}\quad\left(\frac{v_\delta}{v_0}\right)_c&=(1+\lambda_P^2)\sin(\delta+\beta)+\lambda_PK+\frac{\Delta v_\delta}{v_0}\end{aligned}\right\}\tag{18}$$

The calculation of the moment is facilitated by expressing the velocity of the compressible fluid in complex form. Thus, as in the case of an incompressible fluid, a complex velocity is defined in the following way:

$$\frac{1}{v_0}\left(\frac{dw}{dz}\right)_c=\left(-\frac{v_R}{v_0}+i\frac{v_\delta}{v_0}\right)_c e^{-i\delta}$$

Then equation (8) for the moment assumes the Blasius form:

$$\begin{aligned}M_c=&-\frac{1}{2}\rho_0v_0^2\,\text{R. P.}\oint\left(\frac{1}{v_0}\frac{dw}{dz}\right)_c^2\frac{dz}{d\zeta}\zeta dz\\&-\frac{1}{4}\mu\rho_0v_0^2\,\text{R. P.}\oint\left(1-\frac{v_i^2}{v_0^2}\right)\left(\frac{1}{v_0}\frac{dw}{dz}\right)_i^2\frac{dz}{d\zeta}\zeta dz\end{aligned}\tag{19}$$

where the integrals are taken around a circle whose radius R approaches infinity.

Now, from equations (13), (16), (17), and (18), the expression for $1/v_0\,(dw/dz)_c$ may be written as:

$$\begin{aligned}\frac{1}{v_0}\left(\frac{dw}{dz}\right)_c=&e^{i\beta}+\frac{iKr_0}{z}-\frac{r_0^2e^{-i\beta}}{z^2}+\frac{\mu}{4}\Big[(A_0\bar{A}_1e^{-i\beta}+\bar{A}_0A_1e^{i\beta})\frac{1}{z}\\&+\frac{A_0\bar{A}_1e^{i\beta}}{R^2}z-\bar{A}_0A_1R^2e^{i\beta}\frac{1}{z^3}\\&+2(A_1\bar{A}_1e^{-i\beta}-iKr_0\bar{A}_0A_1)\frac{1}{z^2}\log\frac{r_0}{R}+\frac{A_1\bar{A}_1}{R^2}e^{i\beta}\\&+\frac{1}{2}A_1\bar{A}_1e^{-i\beta}\frac{1}{z^2}+iKr_0\frac{A_0\bar{A}_1}{R^2}+iKr_0\bar{A}_0A_1\frac{1}{z^2}\\&+\frac{A_0\bar{A}_2e^{i\beta}z^2}{R^4}-2\bar{A}_0A_2e^{-i\beta}\frac{R^2}{z^4}+3\bar{A}_0A_2e^{i\beta}\frac{1}{z^3}\\&+iK\frac{1}{z^2}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{\bar{A}_nA_{n+1}}{r_0^{2n-1}}\\&-\frac{2e^{-i\beta}}{z^3}\sum_{n=1}^{\infty}\frac{n+1}{n(n+2)}\frac{A_{n+1}\bar{A}_{n+1}}{r_0^{2n}}+\frac{2e^{i\beta}}{z^2}\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{\bar{A}_nA_{n+2}}{r_0^{2n}}\Big]\end{aligned}$$

Also, from equation (12) for $\frac{1}{v_0}\frac{dw}{d\zeta}$:

$$A_0=e^{i\beta},\ A_1=iKr_0,\ A_2=b^2e^{i(\beta+2\gamma)}-r_0^2e^{-i\beta}$$

where $a_1=b^2e^{2i\gamma}$

Therefore

$$\begin{aligned}\frac{1}{v_0}\left(\frac{dw}{dz}\right)_c=&e^{i\beta}+\frac{iKr_0}{z}-\frac{r_0^2e^{-i\beta}}{z^2}+\frac{\mu}{4}\Big[-\frac{iKr_0e^{2i\beta}}{R^2}z\\&+\frac{4K^2r_0^2e^{-i\beta}}{z^2}\log\frac{r_0}{R}+\frac{2K^2r_0^2}{R^2}e^{i\beta}+\frac{b^2e^{i(\beta-2\gamma)}-r_0^2e^{3i\beta}}{R^4}z^2\\&-\frac{iKr_0R^2}{z^3}-\frac{K^2r_0^2e^{-i\beta}}{2z^2}-2R^2\frac{b^2e^{-i(\beta-2\gamma)}-r_0^2e^{-3i\beta}}{z^4}\\&+3\frac{b^2e^{i(\beta+2\gamma)}-r_0^2e^{-i\beta}}{z^3}+\frac{iK}{z^2}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{\bar{A}_nA_{n+1}}{r_0^{2n-1}}\\&-\frac{2e^{-i\beta}}{z^3}\sum_{n=1}^{\infty}\frac{n+1}{n(n+2)}\frac{A_{n+1}\bar{A}_{n+1}}{r_0^{2n}}+\frac{2e^{i\beta}}{z^2}\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{\bar{A}_nA_{n+2}}{r_0^{2n}}\Big]\end{aligned}$$

It may be noted that this expression is not an analytic function of z, since powers of R occur in some of the terms.

The first integral on the right-hand side of equation (19) for M_c then becomes:

$$-\frac{1}{2}\rho_0 v_0^2 \text{ R. P. } \oint\left(\frac{1}{v_0}\frac{dw}{dz}\right)_c^2 \frac{dz}{d\zeta}\zeta dz$$

$$=-\frac{1}{2}\rho_0 v_0^2 \text{ R. P. } \oint\left(\frac{1}{v_0}\frac{dw}{dz}\right)_c^2\left(1+\frac{me^{i\nu}}{z}+\frac{2b^2e^{2i\gamma}}{z^2}+\cdots\right)zdz$$

$$=2\pi b^2\rho_0 v_0^2 \sin 2(\beta+\gamma)+\rho_0 v_0 \Gamma m \cos(\beta+\nu)$$

$$+\frac{\mu}{2}\left[3\pi b^2\rho_0 v_0^2 \sin 2(\beta+\gamma)\right.$$

$$+\frac{1}{2}\rho_0 v_0 \Gamma_i \text{ R. P. } e^{i\beta}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{\overline{A}_n A_{n+1}}{r_0^{2n}}$$

$$\left.-2\pi\rho_0 v_0^2 \text{ R. P. } ie^{2i\beta}\sum_{n=1}^{\infty}\frac{\overline{A}_n A_{n+2}}{(n+1)r_0^{2n}}\right]$$

and the second integral:

$$-\frac{1}{4}\mu\rho_0 v_0^2 \text{ R. P. } \oint\left(1-\frac{v_i^2}{v_0^2}\right)\left(\frac{1}{v_0}\frac{dw}{dz}\right)_i^2\frac{dz}{d\zeta}\zeta dz$$

$$=\frac{\mu}{4}[2\pi b^2\rho_0 v_0^2 \sin 2(\beta+\gamma)+\rho_0 v_0 \Gamma m \cos(\beta+\nu)]$$

Therefore

$$M_c=M_i\left(1+\frac{\mu}{2}\right)+\rho_0 v_0 \Delta\Gamma m \cos(\beta+\nu)$$

$$+\frac{\mu}{2}\left[-\frac{3}{2}\rho_0 v_0 \Gamma_i m \cos(\beta+\gamma)\right.$$

$$+\frac{1}{2}\rho_0 v_0 \Gamma_i \text{ R. P. } e^{i\beta}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{\overline{A}_n A_{n+1}}{r_0^{2n}}$$

$$\left.-2\pi\rho_0 v_0^2 \text{ R. P. } ie^{2i\beta}\sum_{n=1}^{\infty}\frac{\overline{A}_n A_{n+2}}{(n+1)r_0^{2n}}\right] \tag{20}$$

where $M_i=2\pi b^2\rho_0 v_0^2 \sin 2(\beta+\gamma)+\rho_0 v_0 \Gamma_i m \cos(\beta+\nu)$ is the moment due to an incompressible fluid. The circulation Γ consists of the incompressible part Γ_i and the additional circulation $\Delta\Gamma$ related to the additional flow.

Equation (20) for the moment M_c is applicable to arbitrary profiles, but it contains two infinite series that cannot, in general, be replaced by closed forms. In the case of a profile for which the conformal transformation to a circle contains a finite number of terms, the two infinite series may, however, sometimes be replaced by elementary functions. In the following, two such systems of profiles will be discussed, the elliptic and the Joukowski profiles.

THE ELLIPTIC PROFILE

It is well known that the Joukowski transformation

$$\zeta=z+\frac{c^2}{4z}$$

maps the circle of radius $c/2$ with its center at the origin of the z plane into a line segment $(-c, 0; c, 0)$ in the ζ plane. Also, circles concentric with the base circle are transformed into a family of confocal ellipses with common focii at $(-c, 0)$ and $(c, 0)$. If $r_0(>c/2)$ denotes the radius of one of these circles, then the semimajor and the semiminor axes of the ellipse into which it is transformed are, respectively, $r_0+c^2/4r_0$ and $r_0-c^2/4r_0$. The thickness ratio t is then defined as

$$t=\frac{r_0-\dfrac{c^2}{4r_0}}{r_0+\dfrac{c^2}{4r_0}}=\frac{1-\sigma^2}{1+\sigma^2}$$

or

$$\sigma^2=\frac{1-t}{1+t}$$

where

$$\sigma=\frac{c}{2r_0}$$

Now, for the case of the elliptic cylinder, equation (12) becomes:

$$\frac{1}{v_0}\frac{dw}{d\zeta}=\frac{e^{i\beta}+\dfrac{i\Gamma_i}{2\pi v_0}\dfrac{1}{z}-\dfrac{r_0^2e^{-i\beta}}{z^2}}{1-\dfrac{c^2}{4z^2}}$$

$$=\left(e^{i\beta}+\frac{i\Gamma_i}{2\pi v_0}\frac{1}{z}-\frac{r_0^2e^{-i\beta}}{z^2}\right)\sum_0^{\infty}\left(\frac{c^2}{4z^2}\right)^n=\sum_0^{\infty}\frac{A_n}{z^n}$$

or

$$A_0=e^{i\beta},\ A_{2n+1}=iK_i r_0^{2n+1}\sigma^{2n},\ A_{2n}=e^{-i\beta}r_0^{2n}\sigma^{2n-2}(\sigma^2e^{2i\beta}-1)$$

where

$$K_i=\frac{\Gamma_i}{2\pi v_0 r_0}$$

Then

$$e^{i\beta}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{\overline{A}_n A_{n+1}}{r_0^{2n}}=e^{i\beta}\sum_{n=1}^{\infty}\frac{4n+1}{2n(2n+1)}\frac{\overline{A}_{2n}A_{2n+1}}{r_0^{4n}}$$

$$+e^{i\beta}\sum_{n=0}^{\infty}\frac{4n+3}{(2n+1)(2n+2)}\frac{\overline{A}_{2n+1}A_{2n+2}}{r_0^{4n+2}}$$

$$=iK_i r_0(\sigma^2-e^{2i\beta})\sum_{n=1}^{\infty}\left(\frac{1}{2n}+\frac{1}{2n+1}\right)\sigma^{4n-2}$$

$$+iK_i r_0(1-\sigma^2e^{2i\beta})\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+2}\right)\sigma^{4n}$$

and

$$\text{R. P. } e^{i\beta}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{\overline{A}_n A_{n+1}}{r_0^{2n}}$$

$$=K_i r_0 \sin 2\beta\left[\sum_{n=1}^{\infty}\frac{\sigma^{4n-2}}{n}+\frac{1+\sigma^4}{2\sigma^4}\log\frac{1+\sigma^2}{1-\sigma^2}-\frac{1}{\sigma^2}\right]$$

Similarly

$$ie^{2i\beta}\sum_{n=1}^{\infty}\frac{\overline{A}_n A_{n+2}}{(n+1)r_0^{2n}}=ie^{2i\beta}\sum_{n=1}^{\infty}\frac{\overline{A}_{2n}A_{2n+2}}{(2n+1)r_0^{4n}}+ie^{2i\beta}\sum_{n=1}^{\infty}\frac{\overline{A}_{2n-1}A_{2n+1}}{2nr_0^{4n-2}}$$

$$=ie^{2i\beta}r_0^2(1-2\sigma^2\cos 2\beta+\sigma^4)\sum_{n=1}^{\infty}\frac{\sigma^{4n-2}}{2n+1}+ie^{2i\beta}K_i^2r_0^2\sum_{n=1}^{\infty}\frac{\sigma^{4n-2}}{2n}$$

and

$$R.\ P.\ ie^{2i\beta}\sum_{n=1}^{\infty}\frac{\overline{A}_nA_{n+2}}{(n+1)r_0^{2n}}$$
$$=-r_0^2\sin 2\beta(1-2\sigma^2\cos 2\beta+\sigma^4)\left(\frac{1}{2\sigma^4}\log\frac{1+\sigma^2}{1-\sigma^2}-\frac{1}{\sigma^2}\right)$$
$$-K_t^2r_0^2\sin 2\beta\sum_{n=1}^{\infty}\frac{\sigma^{4n-2}}{2n}$$

Therefore, since $m=0$ and $\nu=0$ for an elliptic profile, equation (20) becomes:

$$M_c=M_t\left\{1+\frac{\mu}{2}+\frac{\mu}{2}(1-2\sigma^2\cos 2\beta+\sigma^4)\left(\frac{1}{2\sigma^6}\log\frac{1+\sigma^2}{1-\sigma^2}-\frac{1}{\sigma^4}\right)\right.$$
$$\left.+\frac{\mu}{2}K_t^2\left[\frac{1+\sigma^4}{4\sigma^6}\log\frac{1+\sigma^2}{1-\sigma^2}-\frac{1}{2\sigma^4}-\frac{1}{\sigma^4}\log(1-\sigma^4)\right]\right\} \quad (21)$$

where $M_t=\frac{\pi}{2}\rho_0v_0^2c^2\sin 2\beta$.

For thick profiles, or for small values of σ, this formula may be expressed as a power series in σ. Thus, by making use of the expansions

$$\log\frac{1+\sigma^2}{1-\sigma^2}=2\left(\sigma^2+\frac{1}{3}\sigma^6+\frac{1}{5}\sigma^{10}+\cdots\right)$$

and

$$\log(1-\sigma^4)=-\left(\sigma^4+\frac{1}{2}\sigma^8+\frac{1}{3}\sigma^{12}+\cdots\right)$$

it follows, from equation (21), that

$$M_c=M_t\left[1+\frac{\mu}{2}+\frac{\mu}{2}(1-2\sigma^2\cos 2\beta+\sigma^4)\left(\frac{1}{3}+\frac{1}{5}\sigma^4+\frac{1}{7}\sigma^8\right.\right.$$
$$\left.\left.+\cdots\right)+\frac{\mu}{2}K_t^2\left(\frac{5}{3}+\frac{23}{30}\sigma^4+\frac{53}{105}\sigma^8+\frac{95}{252}\sigma^{12}+\cdots\right)\right]$$

It is seen from this equation that the value of $\frac{M_c-M_t}{M_t}$ for the limiting case of a circle, for which $t=1$ or $\sigma=0$, is:

$$\frac{M_c-M_t}{M_t}=\frac{\mu}{2}\,\frac{4+5K_t^2}{3}$$

On the other hand, equation (21) shows that, for the limiting case of a straight-line segment for which $t=0$ or $\sigma=1$ and the angle of attack β is finite,

$$\frac{M_c-M_t}{M_t}=\infty$$

It is to be noted that, although M_t, and therefore M_c, is zero when the angle of attack β is zero, the ratio $\frac{M_c-M_t}{M_t}$ possesses the limiting value

$$\left(\frac{M_c-M_t}{M_t}\right)_{\beta=0}=\frac{\mu}{2}\left[1+(1-\sigma^2)^2\left(\frac{1}{2\sigma^6}\log\frac{1+\sigma^2}{1-\sigma^2}-\frac{1}{\sigma^4}\right)\right] \quad (22)$$

where $K_t=2\sin\beta$, according to the condition that the rear stagnation point occurs at the end of the major axis of the ellipse. For the two limiting cases of the circle and the straight-line segment, this ratio becomes, respectively:

$$\left(\frac{M_c-M_t}{M_t}\right)_{\substack{\beta=0\\ \sigma=0}}=\frac{4}{3}\,\frac{\mu}{2}$$

and

$$\left(\frac{M_c-M_t}{M_t}\right)_{\substack{\beta=0\\ \sigma=1}}=\frac{\mu}{2} \quad (23)$$

In appendix A, an independent derivation of equation (21), with $K_t=0$, is obtained by a direct integration of the pressures over the surface of an elliptic cylinder. It shows clearly the superiority of the present general method, in which all integrations are performed along a circle at infinity.

It is of interest to investigate the effect of compressibility on the center of pressure. If the rear stagnation point is supposed to be at the end of the major axis, it is immediately seen from the second of equations (18) that, for $\delta=\pi$ and $\lambda_P=1$,

$$\left(\frac{v_\delta}{v_0}\right)_{\delta=\pi}=-2\sin\beta+K_t+\Delta K+\left(\frac{\Delta v_\delta}{v_0}\right)_{\delta=\pi}=0$$

Since $K_t=2\sin\beta$, it follows that the additional circulation ΔK is given by

$$\Delta K=-\left(\frac{\Delta v_\delta}{v_0}\right)_{\delta=\pi}$$

The problem of determining the additional circulation thus amounts to finding an expression for $\Delta v_\delta/v_0$ at the stagnation point $\delta=\pi$. This calculation is given in appendix B and it is shown there that

$$\frac{\Delta\Gamma}{\Gamma_t}=\frac{\Delta K}{K_t}=\frac{\mu}{2}(M+N\sin^2\beta) \quad (24)$$

where M and N are functions of the thickness coefficient t only. Table I presents values of M and N for various values of t.

TABLE I

t	M	N
0	1.00000	∞
.1	1.09808	3.44779
.2	1.19225	2.48066
.3	1.28287	1.98162
.4	1.37029	1.63730
.5	1.45392	1.39813
.6	1.53503	1.20008
.7	1.61342	1.03754
.8	1.68916	.89794
.9	1.76185	.77605
1.0	1.83333	.66667

For the straight-line segment, $\Delta\Gamma/\Gamma_t$ equals infinity if the value of β is other than zero but, for very thin profiles and vanishingly small angles of attack,

$$\frac{\Gamma_t+\Delta\Gamma}{\Gamma_t}=1+\frac{\mu}{2}$$

This result agrees with that given by Glauert (reference 3), namely:

$$\frac{\Gamma_t+\Delta\Gamma}{\Gamma_t}=\frac{1}{\sqrt{1-\mu}}=1+\frac{\mu}{2}+\cdots$$

In reference 3, Glauert has also shown that the lifting force on a body in a compressible fluid is given by

$$L_c=\rho_0v_0\Gamma$$

or

$$L_c=L_t\left(1+\frac{\mu}{2}\right) \quad (25)$$

If c_c and c_i denote, respectively, the centers of pressure in the compressible and the incompressible fluids, then according to equations (23) and (25)

$$\frac{c_c}{c_i}=1$$

That is, for a very thin profile and for vanishingly small angles of attack, the center of pressure is unaffected by the compressibility of the fluid. For the general elliptic profiles, it follows from equations (21) and (24) that

$$\frac{c_c}{c_i}=\frac{1+\frac{\mu}{2}\left\{1+(1-2\sigma^2\cos 2\beta+\sigma^4)\left(\frac{1}{2\sigma^6}\log\frac{1+\sigma^2}{1-\sigma^2}-\frac{1}{\sigma^4}\right)+4\sin^2\beta\left[\frac{1+\sigma^4}{4\sigma^6}\log\frac{1+\sigma^2}{1-\sigma^2}-\frac{1}{2\sigma^4}-\frac{1}{\sigma^4}\log(1-\sigma^4)\right]\right\}}{1+\frac{\mu}{2}(M+N\sin^2\beta)}$$

It may be shown from this expression that, for any given ellipse, there exists an angle of attack β, independent of the stream velocity v_0, to the first order of μ, for which the ratio c_c/c_i equals unity. Furthermore, if the angle of attack is $\left\{\begin{matrix}\text{greater}\\ \text{less}\end{matrix}\right\}$ than this neutral value of β, then c_c/c_i is $\left\{\begin{matrix}\text{greater}\\ \text{less}\end{matrix}\right\}$ than unity and the center of pressure c_c in the compressible fluid is $\left\{\begin{matrix}\text{farther from}\\ \text{nearer to}\end{matrix}\right\}$ the origin than the center of pressure c_i in the incompressible fluid. Table II presents the neutral values of β for the entire range of ellipses including the straight-line profile and the circle.

TABLE II

t	$\cos 2\beta$	$\beta_{neutral}$	
		Degrees	Minutes
0	1.00000	0	0
.1	.98302	5	17
.2	.96199	7	56
.3	.94116	9	53
.4	.92173	11	25
.5	.90348	12	40
.6	.88687	13	46
.7	.87148	14	42
.8	.85965	15	22
.9	.84474	16	11
1.0	.83333	16	47

As a numerical example, consider an elliptic cylinder with a thickness coefficient $t=½$ and with $v_0/c_0=½$. In this case, the neutral angle of attack β is given by $\cos 2\beta=0.90348$, or $\beta=12°40'$. If the angle of attack is increased to 15°, say, then

$$\frac{M_c}{M_i}=1.20776,\ \frac{L_c}{L_i}=1.19345$$

and

$$\frac{c_c}{c_i}=1.012$$

Now

$$c_i=\frac{a}{2}(1-t)$$

Therefore

$$\frac{c_c-c_i}{2a}=0.0015$$

or the motion of the center of pressure away from the origin is 0.15 percent of the chord. Again, if the angle of attack is vanishingly small, then $\cos\beta\cong 1$ and $\sin\beta\cong\beta$, and

$$\frac{M_c}{M_i}=1.14486,\ \frac{L_c}{L_i}=1.18174$$

Then

$$\frac{c_c-c_i}{2a}=-0.0039$$

or the motion of the center of pressure toward the origin is 0.39 percent of the chord. In general, when the angle of attack β is vanishingly small, the center of pressure c_c in the compressible fluid is nearer to the origin than the center of pressure c_i in the incompressible fluid. Table III gives values for $\left(\frac{c_c-c_i}{2a}\right)_{\beta=0}$ for various values of t and for the critical values of μ.[3]

TABLE III

t	μ_{crit}	$\left(\frac{c_c}{c_i}\right)_{\beta=0}$	$\left(\frac{c_c-c_i}{2a}\right)_{\beta=0}$
0	1.00000	1.00000	0
.1	.73445	.97960	−.00459
.2	.57608	.96988	−.00602
.3	.46077	.96459	−.00620
.4	.39514	.96155	−.00577
.5	.33293	.96038	−.00495
.6	.28944	.95971	−.00403
.7	.25120	.96003	−.00300
.8	.22212	.96087	−.00198
.9	.19714	.96118	−.00097
1.0	.17610	.96204	0

THE SYMMETRICAL JOUKOWSKI PROFILE

The Joukowski profiles are derived by means of the conformal transformation

$$\zeta=z'+\frac{a^2}{z'}$$

which maps the circle of radius a with its center at the origin of the z' plane into a line segment $(-2a, 0; 2a, 0)$ in the ζ plane. Any other circle of radius $r_0(>a)$ with its center lying on the real axis at a distance $a\epsilon$ from O (see fig. 4) and touching the base circle at $(-a, 0)$ is mapped into a symmetrical Joukowski profile with a sharp trailing edge. The variables z and z' are connected by the equation

$$z'=\epsilon a+z$$

The circulation Γ_i is chosen in accordance with the Kutta-Joukowski hypothesis that the rear stagnation point of the flow occurs at the point A of the circle. In

[3] The critical value of μ is defined as that value of μ at which the maximum velocity of the fluid at the surface of the obstacle just attains the local velocity of sound. A list of such values of μ for a set of elliptic cylinders with the angle of attack equal to zero is given in table IV of reference 4.

terms of the complex coordinate z, the potential function of the flow past the circle is

$$w=v_0\left(ze^{i\beta}+\frac{r_0^2}{ze^{i\beta}}\right)+\frac{i\Gamma_i}{2\pi}\log\frac{ze^{i\beta}}{r_0}$$

The complex velocity at the point A is then given by

$$\frac{dw}{dz}=0=v_0(e^{i\beta}-e^{-i\beta})-\frac{i\Gamma_i}{2\pi r_0}$$

Therefore

$$\Gamma_i=4\pi v_0 r_0\sin\beta \text{ or } K_i=2\sin\beta$$

where

$$I=\text{R. P. } i\frac{1-e^{2i\beta}}{2\sin 2\beta}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{\bar{A}_nA_{n+1}}{r_0^{2n+1}}$$

$$J=-\text{R. P.}\frac{ie^{2i\beta}}{\sin 2\beta}\sum_{n=1}^{\infty}\frac{\bar{A}_nA_{n+2}}{(n+1)r_0^{2n+2}}$$

and

$$M_i=2\pi\rho_0v_0^2a^2\,[1+\epsilon(1+\epsilon)]\sin 2\beta$$

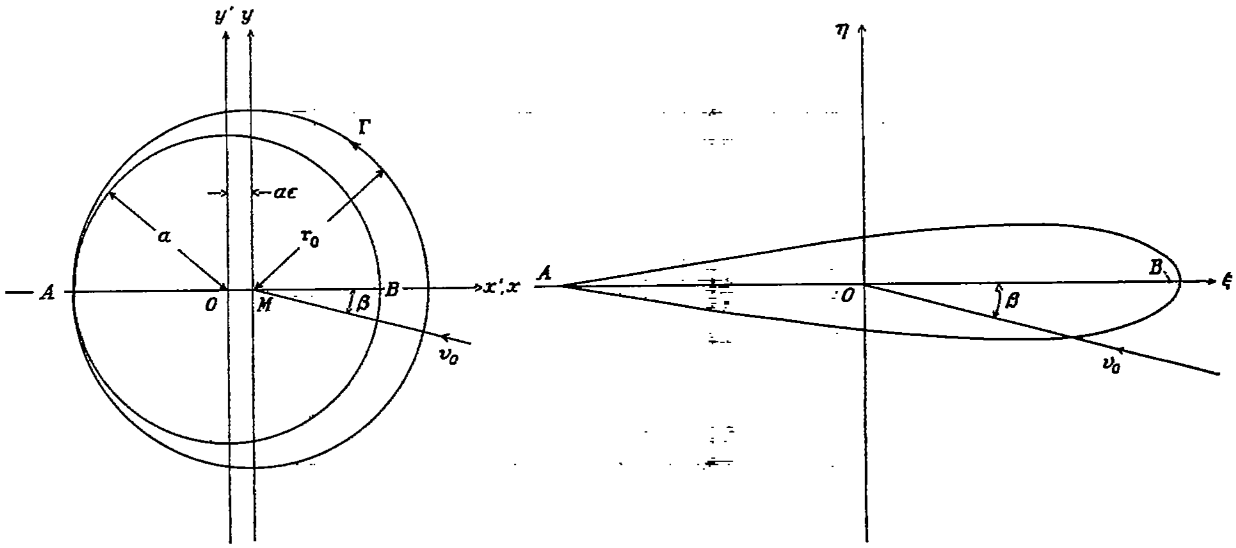

FIGURE 4.—Transformation of a symmetrical Joukowski profile into a circle.

The complex velocity in the ζ plane is given by

$$\frac{dw}{d\zeta}=\frac{dw}{dz}\frac{dz}{dz'}\frac{dz'}{d\zeta}=v_0\frac{(z+r_0)(ze^{i\beta}-r_0e^{-i\beta})z'^2}{z^2(z'+a)(z'-a)}$$

Then, since

$$z'+a=z+a(1+\epsilon)=z+r_0 \text{ and } z'-a=z-a(1-\epsilon)$$

it follows that

$$\frac{1}{v_0}\frac{dw}{d\zeta}=\frac{\left(1+\frac{\epsilon a}{z}\right)^2\left(e^{i\beta}-\frac{r_0}{z}e^{-i\beta}\right)}{1-\frac{a(1-\epsilon)}{z}}$$

Putting $\frac{\epsilon}{1+\epsilon}=h$ and $\frac{1-\epsilon}{1+\epsilon}=k$,

$$\frac{1}{v_0}\frac{dw}{d\zeta}=\sum_{n=0}^{\infty}\frac{A_n}{z^n}=e^{i\beta}\sum_{n=0}^{\infty}k^n\left(\frac{r_0}{z}\right)^n+(2he^{i\beta}-e^{-i\beta})\sum_{n=0}^{\infty}k^n\left(\frac{r_0}{z}\right)^{n+1}$$
$$+(h^2e^{i\beta}-2he^{-i\beta})\sum_{n=0}^{\infty}k^n\left(\frac{r_0}{z}\right)^{n+2}-h^2e^{-i\beta}\sum_{n=0}^{\infty}k^n\left(\frac{r_0}{z}\right)^{n+3}$$

Therefore

$$A_0=e^{i\beta},\ A_1=2ir_0\sin\beta,\ A_2=a^2e^{i\beta}-r_0^2e^{-i\beta},$$

and

$$A_n=a^2r_0^{n-2}k^{n-3}(ke^{i\beta}-e^{-i\beta}) \text{ for } n\geqq 3$$

Equation (20) giving the moment M_c then may be written as

$$\frac{M_c-M_i}{M_i}=\frac{\epsilon(1+\epsilon)}{1+\epsilon(1+\epsilon)}\frac{\Delta\Gamma}{\Gamma_i}$$
$$+\frac{\mu}{2}\left[1-\frac{3}{2}\frac{\epsilon(1+\epsilon)}{1+\epsilon(1+\epsilon)}+\frac{(1+\epsilon)^2}{1+\epsilon(1+\epsilon)}(I+J)\right] \quad (26)$$

In appendix C, closed expressions are derived for I and J. They may be written as follows:

$$(1+\epsilon)^2(I+J)=\frac{(1+k)^2}{8k^5}(1-2k\cos 2\beta+k^2)$$
$$\left[\frac{(1+k)^2}{k}\log\frac{1}{1-k^2}-\left(1+2k+\frac{3}{2}k^2+k^3+\frac{5}{6}k^4+\frac{2}{3}k^5\right)\right]$$
$$+\frac{1}{6}(3+2k)(1-2k\cos 2\beta+k^2)+\frac{1}{6}(9+5k)(1-\cos 2\beta)$$
$$+\frac{1}{2}k(1-k)+\frac{1}{48}(15+5k+4k^2)(1-k)^2 \quad (27)$$

For small values of k, or thick profiles,

$$(1+\epsilon)^2(I+J)=\frac{k(1+k)^2}{4}(1-2k\cos 2\beta+k^2)\left(\frac{7}{24}+\frac{1}{4}k\right.$$
$$\left.+\frac{9}{40}k^2+\frac{1}{5}k^3+\frac{11}{60}k^4+\frac{1}{6}k^5+\ldots\right)$$
$$+\frac{1}{6}(3+2k)(1-2k\cos 2\beta+k^2)+\frac{1}{6}(9+5k)(1-\cos 2\beta)$$
$$+\frac{1}{2}k(1-k)+\frac{1}{48}(15+5k+4k^2)(1-k)^2 \quad (28)$$

It remains to evaluate the ratio of the additional circulation $\Delta\Gamma$ and the circulation Γ_i of the incompressible fluid. This ratio may be obtained from equation

(22) of reference 2 by evaluating $\Delta v_\delta/v_0$ for $\delta=\pi$ and is written as follows (see appendix D):

$$\frac{\Delta\Gamma}{\Gamma_i}=-\frac{1}{2\sin\beta}\left(\frac{\Delta v_\delta}{v_0}\right)_{\delta=\pi}=\frac{\mu}{2}(M+N\sin^2\beta) \tag{29}$$

where M and N depend only on the thickness coefficient ϵ of the profile. Table IV lists values of M and N for several values of ϵ.

TABLE IV

ϵ	M	N
0	1.00000	∞
.03	1.03242	3.76210
.05	1.05496	3.73331
.10	1.11153	2.88585
.20	1.21659	2.17368
.30	1.32617	1.80592
.40	1.40925	1.56348
.50	1.49044	1.38092

It is interesting to note that $M=1$ for a straight-line profile but that N is infinite on account of a term containing log ϵ. For very thin profiles and vanishingly small angles of attack, however,

$$\frac{\Gamma_i+\Delta\Gamma}{\Gamma_i}=1+\frac{\mu}{2} \tag{30}$$

This result agrees with Glauert's well-known formula (see section on the elliptic cylinder)

$$\frac{\Gamma_i+\Delta\Gamma}{\Gamma_i}=\frac{1}{\sqrt{1-\mu}}=1+\frac{\mu}{2}+\ .\ .\ .$$

It follows from equations (26) and (27) that, for an infinitely thin profile with an angle of attack so small that $\sin\beta\cong\beta$ and $\cos\beta\cong1$,

$$\frac{M_c}{M_i}=1+\frac{\mu}{2} \tag{31}$$

With $L_c=\rho_0v_0(\Gamma_i+\Delta\Gamma)$ and $L_i=\rho_0v_0\Gamma_i$, it therefore follows from equations (30) and (31) that

$$\frac{M_c}{L_c}=\frac{M_i}{L_i}$$

or

$\frac{c_c}{c_i}=1$ (as in the case of the ellipse). In general, according to equations (26) and (29),

$$\frac{c_c}{c_i}=\frac{1+\frac{\mu}{2}\left[1+\frac{\epsilon(1+\epsilon)}{1+\epsilon(1+\epsilon)}\left(-\frac{3}{2}+M+N\sin^2\beta\right)+\frac{(1+\epsilon)^2}{1+\epsilon(1+\epsilon)}(I+J)\right]}{1+\frac{\mu}{2}(M+N\sin^2\beta)} \tag{32}$$

Again, as in the case of elliptic cylinders, a neutral value for β is obtained when the centers of pressure in the compressible and the incompressible fluids coincide. It may be shown by means of numerical examples that, when the angle of attack is less than the neutral value of β, the center of pressure c_c in the compressible fluid moves from its position c_i in the incompressible fluid toward the origin and that, when the angle of attack is greater than the neutral value of β, the center of pressure c_c moves toward the nose of the profile.

In order to gain some idea as to the order of magnitude of the movement of the center of pressure due to compressibility, consider the case of a thin profile, say $\epsilon=0.05$. Then equation (32) becomes

$$\frac{c_c}{c_i}=\frac{1+\frac{\mu}{2}(7.00142-5.95991\cos2\beta)}{1+\frac{\mu}{2}(2.92162-1.86666\cos2\beta)}$$

Letting c_c and c_i coincide, i. e., $c_c/c_i=1$, yields

$$\cos2\beta=0.99671$$

or a neutral angle

$$\beta=2°20'$$

If the angle of attack is taken to be 0°, then

$$\frac{c_c}{c_i}=\frac{1+0.52075\ \mu}{1+0.52748\ \mu}$$

which shows that c_c is nearer the origin than c_i. It is seen that the movement of the center of pressure is very small. Thus, even for a large value of v_0/c_0, say 0.70, the center of pressure moves only about 0.07 percent of the chord toward the origin.

Again, if the angle of attack is increased to 4°,

$$\frac{c_c}{c_i}=\frac{1+0.54975\ \mu}{1+0.53656\ \mu}$$

which shows that c_c is nearer the nose of the profile than c_i. The center of pressure in this case, with $v_0/c_0=0.70$, moves about 0.13 percent of the chord toward the nose.

This numerical example indicates that, although the effect of compressibility on the lift and the moment of a thin airfoil may be large, namely

$$\frac{L_c-L_i}{L_i}=0.25517,\ \frac{M_c-M_i}{M_i}=0.25847$$

for

$$\beta=0°,\ \frac{v_0}{c_0}=0.70$$

and

$$\frac{L_c-L_i}{L_i}=0.26938,\ \frac{M_c-M_i}{M_i}=0.26291$$

for

$$\beta=4°,\ \frac{v_0}{c_0}=0.70$$

the effect on the center of pressure may be considered negligible.

As an example of a thicker airfoil, let $\epsilon=0.10$. The thickness of Joukowski airfoils is proportional to ϵ and a value of ϵ of 0.10 gives a maximum thickness of about 0.13 times the chord, a value rarely exceeded in practice. For this case, equation (32) becomes:

$$\frac{c_c}{c_i}=\frac{1+\frac{\mu}{2}(5.43168-4.34449\cos 2\beta)}{1+\frac{\mu}{2}(2.55446-1.44293\cos 2\beta)}$$

Putting $c_c/c_i=1$ yields

$$\cos 2\beta=0.99161$$

or a neutral angle

$$\beta=3°42'$$

If the angle of attack is taken to be 0° and $v_0/c_0=0.70$, then

$$\frac{c_c}{c_i}=\frac{1.26636}{1.27232}=0.99531$$

or the center of pressure is displaced toward the origin a distance equal to 0.12 percent of the chord.

If the angle of attack is increased to 6° with $v_0/c_0=0.70$, then

$$\frac{c_c}{c_i}=\frac{1.28962}{1.28005}=1.00748$$

or the center of pressure is displaced toward the nose of the airfoil a distance equal to 0.19 percent of the chord.

CONCLUDING REMARKS

It has been shown that, for a thin airfoil and for small angles of attack, the moment is given by equation (31):

$$M_c=M_i\left(1+\frac{\mu}{2}\right)$$

This result is, however, limited to small values of μ. On the other hand, Glauert (reference 3) has shown that the lift on an airfoil in a compressible fluid is given by

$$L_c=\frac{L_i}{\sqrt{1-\mu}}=L_i\left(1+\frac{\mu}{2}+\ \cdot\ \cdot\ \cdot\right)$$

The validity of this formula depends on the assumption that the velocity at the surface of the airfoil does not differ appreciably from the undisturbed velocity v_0, although the velocity in the undisturbed stream may be large. This assumption means that the airfoil must be thin and the angle of attack small. The similarity of the two preceding formulas, insofar as the first power of μ is concerned, strongly suggests that the expression for the moment on a thin airfoil at small angles of attack and for large stream velocities is given by:

$$M_c=\frac{M_i}{\sqrt{1-\mu}} \tag{33}$$

This result is indeed implied in Glauert's work, where it is stated that the lift distribution along the chord remains unaltered but that the strength of each elementary vortex is increased by the factor $(1-\mu)^{-\frac{1}{2}}$.

In view of this discussion, it would appear that a more accurate expression for the moment M_c may be obtained by introducing the factor $(1-\mu)^{-\frac{1}{2}}$ into equation (20). Thus, equation (20) is replaced by

$$\begin{aligned}M_c=\frac{1}{\sqrt{1-\mu}}\Bigg\{&M_i+\rho_0v_0\Delta\Gamma m\cos(\beta+\sigma)\\&+\frac{\mu}{2}\Bigg[-\frac{3}{2}\rho_0v_0\Gamma_i m\cos(\beta+\sigma)\\&+\frac{1}{2}\rho_0v_0\Gamma_i\text{ R. P. }e^{i\beta}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{\overline{A}_n\overline{A}_{n+1}}{r_0^{2n}}\\&-2\pi\rho_0v_0^2\text{ R. P. }ie^{2i\beta}\sum_{n=1}^{\infty}\frac{\overline{A}_nA_{n+2}}{(n+1)r_0^{2n}}\Bigg]\Bigg\}\end{aligned} \tag{34}$$

This formula differs from equation (20) in that it is valid for large stream velocities and differs from equation (33) in that it estimates the effect of large disturbances to the main flow v_0 because of the presence of an obstacle.

As an example, consider the case of symmetrical Joukowski airfoils with sharp trailing edges. Equation (26) is then replaced by

$$\begin{aligned}M_c=\frac{M_i}{\sqrt{1-\mu}}\Bigg\{&1+\frac{\epsilon(1+\epsilon)}{1+\epsilon(1+\epsilon)}\frac{\Delta\Gamma}{\Gamma_i}\\&+\frac{\mu}{2}\left[-\frac{3}{2}\frac{\epsilon(1+\epsilon)}{1+\epsilon(1+\epsilon)}+\frac{(1+\epsilon)^2}{1+\epsilon(1+\epsilon)}(I+J)\right]\Bigg\}\end{aligned} \tag{35}$$

In an analogous manner, the formula for the lift becomes

$$L_c=\frac{L_i}{\sqrt{1-\mu}}\left[1+\frac{\mu}{2}(-1+M+N\ \sin^2\beta)\right] \tag{36}$$

As a numerical example, consider an airfoil for which $\epsilon=0.10$. For this case

$$M=1.11153$$
$$N=2.88585$$
$$\frac{\Delta\Gamma}{\Gamma_i}=\frac{\mu}{2}(1.11153+2.88585\sin^2\beta)$$

and

$$(1+\epsilon)^2(I+J)=4.80317-4.66366\cos 2\beta$$

Table V gives the values of $\frac{2}{\mu}\frac{\Delta\Gamma}{\Gamma_i}$ and of $(1+\epsilon)^2(I+J)$ for several angles of attack.

TABLE V

β (deg.)	$\frac{2}{\mu}\frac{\Delta\Gamma}{\Gamma_i}$	$(1+\epsilon)^2(I+J)$
0	1.11153	0.13951
2	1.11505	.15039
4	1.12557	.18489
6	1.14306	.21111
8	1.16743	.32018
10	1.19355	.42078

Table VI lists the values of the ratios M_c/M_i and L_c/L_i as given by equations (35) and (36), respectively, for various values of v_0/c_0. The last column gives the values of M_c/M_i (or L_c/L_i) calculated from equation (33).

TABLE VI

v_0/c_0 \ β (deg.)	M_c/M_i: 0	2	4	6	8	10	L_c/L_i: 0	2	4	6	8	10	$\frac{1}{\sqrt{1-\mu}}$
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
.40	1.09871	1.09964	1.10240	1.10700	1.11340	1.12165	1.10084	1.10114	1.10206	1.10359	1.10572	1.10843	1.09110
.50	1.16732	1.16885	1.17342	1.18102	1.19168	1.20513	1.17083	1.17134	1.17256	1.17538	1.17890	1.18339	1.15478
.60	1.26962	1.27200	1.27913	1.29098	1.30749	1.32857	1.27509	1.27589	1.27825	1.28219	1.28767	1.29467	1.25000
.70	1.43005	1.43383	1.44469	1.46276	1.48794	1.52008	1.43854	1.43975	1.44336	1.44936	1.45772	1.46840	1.40028
.75	1.54896	1.55347	1.56693	1.58933	1.62053	1.66038	1.55932	1.56081	1.56529	1.57272	1.58308	1.59632	1.51189
.80	1.71317	1.71882	1.73571	1.76379	1.80293	1.85291	1.72615	1.72803	1.73364	1.74297	1.75596	1.77256	1.66667
.85	1.95812	1.96538	1.98710	2.02821	2.07353	2.13779	1.97480	1.97722	1.98443	1.99643	2.01814	2.03448	1.89632

Figures 5 and 6 show graphs based on table VI with M_c/M_i and L_c/L_i, respectively, as ordinates and the Mach number v_0/c_0 as the abscissa. The dashed curves represent the Glauert approximation

$$\frac{M_c}{M_i}=\frac{L_c}{L_i}=\frac{1}{\sqrt{1-\mu}}$$

It is seen from an examination of the table and the curves that, below the neutral angle of attack (in this case 3°42′), the Glauert approximation is better for the moment than for the lift but that, above this angle, the approximation is better for the lift than for the mo-

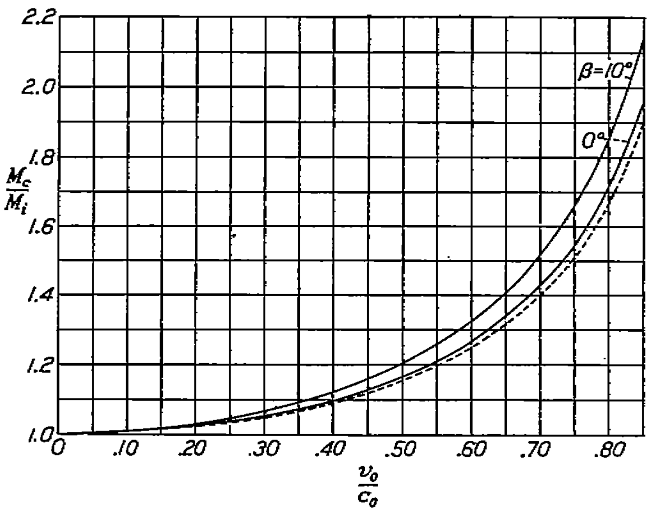

FIGURE 5.—The variation of the ratio M_c/M_i with the Mach number v_0/c_0.

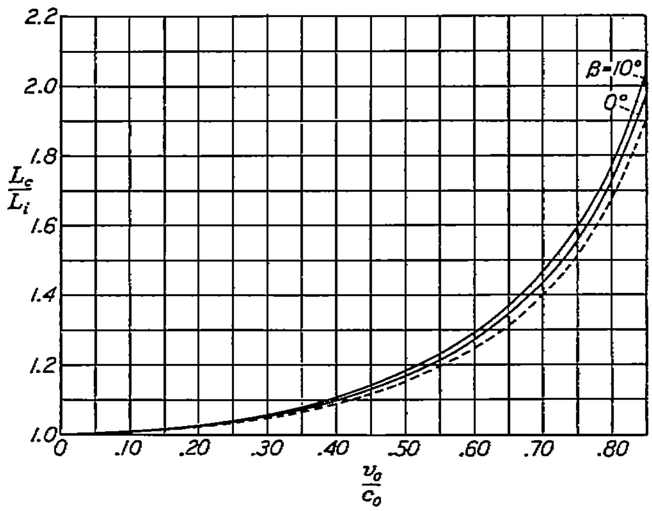

FIGURE 6.—The variation of the ratio L_c/L_i with the Mach number v_0/c_0.

ment. In any case, it appears that, at least for thin airfoils and for small angles of attack, the Glauert approximation for both the moment and the lift is sufficiently good. It then follows that the effect of compressibility on the center of pressure is negligible.

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *March 24, 1939.*

APPENDIX A

EFFECT OF COMPRESSIBILITY ON THE COUPLE ABOUT AN ELLIPTIC CYLINDER

The moment on a body due to the fluid motion is, according to equation (1), given by

$$M=-\int_l [\mathbf{r}\ \mathbf{n}]\, p\, ds$$

where the positive direction of the normal vector $\mathbf{n}$ is taken from the body into the fluid. The components of the vectors $\mathbf{r}$ and $\mathbf{n}$ are, respectively, (x, y) and $(\cos \mathbf{n}x, \cos \mathbf{n}y)$.

Therefore

$$[\mathbf{r}\ \mathbf{n}]=x \cos \mathbf{n}y-y \cos \mathbf{n}x$$

From figure 1, it is seen that

$$dx=ds \cos \mathbf{n}y \text{ and } dy=-ds \cos \mathbf{n}x$$

Hence

$$M=-\oint p(x\, dx+y\, dy)=-\frac{1}{2}\oint p\, dr^2$$

Now, the equations of transformation from Cartesian to elliptic coordinates are:

$$x=c \cosh \xi \cos \eta,\ y=c \sinh \xi \sin \eta$$

or

$$z=c \cosh (\xi+i\eta)$$

where ξ takes on all values from zero to infinity and η takes on all values from 0 to 2π. Then ξ=constant and η=constant represent confocal ellipses and hyperbolas, respectively, the distance between the foci being $2c$.

For any given ellipse $\xi=\alpha$, say,

$$x=c \cosh \alpha \cos \eta,\ y=c \sinh \alpha \sin \eta$$

or

$$r^2=c^2 (\cosh^2 \alpha \cos^2 \eta+\sinh^2 \alpha \sin^2 \eta)$$

Then

$$dr^2=-c^2 \sin 2\eta\, d\eta$$

Therefore

$$M=-\frac{1}{2}c^2\int_0^{2\pi} p \sin 2\eta\, d\eta$$

Now, according to equation (7), the pressure p in a compressible fluid is given by

$$p=\text{constant}\ \frac{1}{2}\rho_0 v_c^2-\frac{1}{4}\mu\rho_0 v_i^2+\frac{1}{8}\mu\frac{\rho_0 v_i^4}{v_0^2}+\ .\ .\ .$$

so that

$$M_c=\frac{1}{4}\rho_0 c^2\int_0^{2\pi} v_c^2 \sin 2\eta d\eta+\frac{1}{8}\mu\rho_0 c^2\int_0^{2\pi} v_i^2 \sin 2\eta d\eta$$

$$-\frac{1}{16}\mu\frac{\rho_0 c^2}{v_0^2}\int_0^{2\pi} v_i^4 \sin 2\eta d\eta$$

The last two integrals in this expression are easily obtained since they involve a knowledge only of the velocity v_i in the incompressible fluid. Thus, suppose the elliptic cylinder to be in a flow of velocity v_0 inclined at an angle β to the negative direction of the real axis and the circulation to be taken as zero. Then, if r_0 is the radius of the circle into which the ellipse $\zeta=\alpha$ is mapped by the Joukowski transformation,

$$z=z'+\frac{c^2}{4z'}$$

and the complex potential of the flow past the circle is

$$w=v_0\left(z'e^{i\beta}+\frac{r_0^2}{z'e^{i\beta}}\right)$$

it follows that the complex velocity is given by

$$\frac{1}{v_0}\frac{dw}{dz}=\frac{e^{i\beta}-\dfrac{r_0^2}{z'^2e^{i\beta}}}{1-\dfrac{c^2}{4z'^2}}$$

On the surface of the cylinder, $z'=r_0e^{i\theta}$. Therefore,

$$\frac{1}{v_0}\frac{dw}{dz}=\frac{e^{i\beta}-e^{-i(2\theta+\beta)}}{1-\dfrac{c^2}{4r_0^2}e^{-2i\theta}}$$

and

$$\frac{v_i^2}{v_0^2}=2\frac{1-\cos 2(\theta+\beta)}{1-2\sigma^2 \cos 2\theta+\sigma^4}$$

where

$$\sigma=\frac{c}{2r_0}$$

Now, from the Joukowski transformation,

$$z'=\frac{z+\sqrt{z^2-c^2}}{2}$$

where the positive sign of the radical has been chosen in order that the regions at infinity of the z and z' planes shall coincide.

Then if (x, y) is replaced by the elliptic coordinates (ξ, η), i. e., $z=c \cosh (\xi+i\eta)$, it follows that:

$$z'=\frac{c}{2}e^{\xi+i\eta}$$

or, on the surface of the cylinder,

$$r_0e^{i\theta}=\frac{c}{2}e^{\alpha}e^{i\eta}$$

Therefore

$$\sigma=e^{-\alpha} \text{ and } \theta=\eta$$

and

$$\frac{v_i^2}{v_0^2}=2\frac{1-\cos 2(\eta+\beta)}{1-2e^{-2\alpha} \cos 2\eta+e^{-4\alpha}}=\frac{e^{2\alpha}[1-\cos 2(\eta+\beta)]}{\cosh 2\alpha-\cos 2\eta}$$

With this expression for v_i^2/v_0^2, it is very easy to calculate the last two integrals in the expression for M_c. Thus

$$\frac{1}{8}\mu\rho_0 c^2\int_0^{2\pi} v_i^2 \sin 2\eta d\eta$$
$$=\frac{1}{8}\mu\rho_0 v_0^2 c^2 e^{2\alpha}\int_0^{2\pi}\frac{1-\cos 2(\eta+\beta)}{\cosh 2\alpha-\cos 2\eta}\sin 2\eta d\eta$$

Now

$$\int_0^{2\pi}\frac{\sin 2n\eta\, d\eta}{\cosh 2\alpha-\cos 2\eta}=0$$

and

$$\int_0^{2\pi}\frac{\cos 2n\eta\, d\eta}{\cosh 2\alpha-\cos 2\eta}=2\pi e^{-2n\alpha}\operatorname{csch} 2\alpha$$

Therefore

$$\frac{1}{8}\mu\rho_0 c^2\int_0^{2\pi} v_i^2 \sin 2\eta d\eta=\frac{1}{4}\mu\pi\rho_0 v_0^2 c^2 \sin 2\beta$$

Similarly,

$$\int_0^{2\pi}\frac{\sin 2n\eta\, d\eta}{(\cosh 2\alpha-\cos 2\eta)^2}=0$$

and

$$\int_0^{2\pi}\frac{\cos 2n\eta\, d\eta}{(\cosh 2\alpha-\cos 2\eta)^2}=2\pi e^{-2n\alpha}\operatorname{csch}^2 2\alpha\,(\coth 2\alpha+n)$$

Hence,

$$-\frac{1}{16}\frac{\mu\rho_0 c^2}{v_0^2}\int_0^{2\pi} v_i^4 \sin 2\eta d\eta$$
$$=-\frac{1}{16}\mu\rho_0 v_0^2 c^2 e^{4\alpha}\int_0^{2\pi}\frac{[1-\cos 2(\eta+\beta)]^2}{(\cosh 2\alpha-\cos 2\eta)^2}\sin 2\eta d\eta$$
$$=\frac{1}{2}\mu\pi\rho_0 v_0^2 c^2\frac{e^{2\alpha}-\cos 2\beta}{e^{-2\alpha}-e^{2\alpha}}\sin 2\beta$$

The sum of the two integrals becomes simply

$$-\frac{1}{4}\mu\pi\rho_0 v_0^2 c^2\frac{\cosh 2\alpha-\cos 2\beta}{\sinh 2\alpha}\sin 2\beta$$

It is much more difficult to calculate the first integral in the expression for M_c, for it is necessary in this case to know the velocity v_c at the surface of the elliptic cylinder in a compressible fluid. For this purpose, it is convenient to make use of equation (13) of reference 2. This equation, when applied to the case of an elliptic profile, becomes

$$\frac{\Delta v}{v_0}=\frac{\mu}{2}\Bigg\{\frac{\cos 2\beta-\sigma^2}{\sigma^2}\sin(\eta+\beta)$$
$$+\sum_{n=0}^{\infty}(2n+1)\sin[(2n+1)\eta+\beta]\int_0^1\lambda^{2n+1}a_{2n}d\lambda$$
$$-\sum_{n=0}^{\infty}(2n+1)\sin[(2n+1)\eta-\beta]\int_0^1\lambda^{2n-1}a_{2n+2}d\lambda$$
$$+4\sin\beta\sum_{n=1}^{\infty}n\cos 2n\eta\int_0^1\lambda^{2n-1}a_{2n}d\lambda$$
$$-\sin 2\beta\sum_{n=1}^{\infty}\sigma^{2n-2}\cos[(2n+1)\eta+\beta]$$
$$+\sin 2\beta\sum_{n=1}^{\infty}\sigma^{2n-2}\cos[(2n-1)\eta-\beta]$$
$$+2\sin\beta\sin 2\beta\sum_{n=1}^{\infty}\sigma^{2n-2}\sin 2n\eta\Bigg\}$$

where $\lambda=r_0/r$, $\sigma=c/2r_0$, and the a_{2n}'s are the coefficients of the cosine terms in the Fourier development of v_i^2/v_0^2 in the region of flow. For the elliptic cylinder

$$\frac{v_i^2}{v_0^2}=\frac{1-2\lambda^2\cos 2(\theta+\beta)+\lambda^4}{1-2\sigma^2\lambda^2\cos 2\theta+\sigma^4\lambda^4}$$
$$=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_{2n}\cos 2n\theta+b_{2n}\sin 2n\theta)$$

so that

$$a_0=2\frac{1+\lambda^4(1-2\sigma^2\cos 2\beta)}{1-\sigma^4\lambda^4}$$
$$a_{2n}=2\frac{\sigma^2(1+\lambda^4)-(1+\sigma^4\lambda^4)\cos 2\beta}{\sigma^2(1-\sigma^4\lambda^4)}(\sigma\lambda)^{2n}$$
$$b_{2n}=\frac{2}{\sigma^2}(\sigma\lambda)^{2n}\sin 2\beta$$

Now, it is recalled that $\Delta v/v_0$ refers to the plane of the circle into which the elliptic profile is mapped by the Joukowski transformation. Therefore

$$\frac{v_c^2}{v_0^2}=\frac{1}{1-2\sigma^2\cos 2\eta+\sigma^4}\left[2\sin(\eta+\beta)+\frac{\Delta v}{v_0}\right]$$

or

$$\frac{v_c^2}{v_0^2}\sin 2\eta=\frac{\sin 2\eta}{1-2\sigma^2\cos 2\eta+\sigma^4}\left[2\sin(\eta+\beta)+\frac{\Delta v}{v_0}\right]$$

But

$$\frac{\sin 2\eta}{1-2\sigma^2\cos 2\eta+\sigma^4}=\sum_{n=1}^{\infty}\sigma^{2n-2}\sin 2n\eta$$

Therefore, the first integral in the expression for M_c becomes

$$\frac{1}{4}\rho_0 v_0^2 c^2\int_0^{2\pi}\frac{v_c^2}{c_0^2}\sin 2\eta d\eta=\frac{1}{2}\pi\rho_0 v_0^2 c^2\sin 2\beta\Bigg\{1+\frac{\mu}{4}\Bigg[\frac{\cos 2\beta-\sigma^2}{\sigma^2}$$
$$+2\frac{1-\sigma^2\cos 2\beta}{1-\sigma^4}+\sum_{n=1}^{\infty}(2n-1)\sigma^{2n-2}\int_0^1\lambda^{2n-1}a_{2n-2}d\lambda$$
$$-\sum_{n=1}^{\infty}(2n+1)\sigma^{2n-2}\int_0^1\lambda^{2n-1}a_{2n+2}d\lambda\Bigg]\Bigg\}$$

The last two integrals in this expression may be written as

$$2\sum_{n=1}^{\infty}n\sigma^{2n-2}\int_0^1\lambda^{2n-1}(a_{2n-2}-a_{2n+2})d\lambda$$
$$-\sum_{n=1}^{\infty}\sigma^{2n-2}\int_0^1\lambda^{2n-1}(a_{2n-2}+a_{2n+2})d\lambda$$
$$=2\sum_{n=1}^{\infty}n\sigma^{4n-6}\int_0^1[\sigma^2(1+\lambda^4)-(1+\sigma^4\lambda^4)\cos 2\beta]\lambda^{4n-2}d\lambda^2$$
$$-\sum_{n=1}^{\infty}\sigma^{4n-6}\int_0^1\frac{\sigma^2(1+\lambda^4)-(1+\sigma^4\lambda^4)\cos 2\beta}{1-\sigma^4\lambda^4}(1+\sigma^4\lambda^4)\lambda^{4n-2}d\lambda^2$$
$$=2\int_0^1\frac{\sigma^2(1+\lambda^4)-(1+\sigma^4\lambda^4)\cos 2\beta}{\sigma^2(1-\sigma^4\lambda^4)^2}d\lambda^2$$
$$-\int_0^1\frac{\sigma^2(1+\lambda^4)-(1+\sigma^4\lambda^4)\cos 2\beta}{\sigma^2(1-\sigma^4\lambda^4)^2}(1+\sigma^4\lambda^4)d\lambda^2$$
$$=\int_0^1\frac{\sigma^2(1+\lambda^4)-(1+\sigma^4\lambda^4)\cos 2\beta}{\sigma^2(1-\sigma^4\lambda^4)}d\lambda^2=-\frac{1-\sigma^2\cos 2\beta}{\sigma^4}$$
$$+\frac{1-2\sigma^2\cos 2\beta+\sigma^4}{2\sigma^6}\log\frac{1+\sigma^2}{1-\sigma^2}$$

Therefore

$$\frac{1}{4}\rho_0 v_0^2 c^2 \int_0^{2\pi} \frac{v_c^2}{v_0^2} \sin 2\eta d\eta = \frac{\pi}{2}\rho_0 v_0^2 c^2 \sin 2\beta \left\{1+\frac{\mu}{2}\left[\frac{\cos 2\beta - \sigma^2}{\sigma^2} + 2\frac{1-\sigma^2 \cos 2\beta}{1-\sigma^4} - \frac{1-\sigma^2\cos 2\beta}{\sigma^4} + \frac{1-2\sigma^2\cos 2\beta+\sigma^4}{2\sigma^6}\log\frac{1+\sigma^2}{1-\sigma^2}\right]\right\}$$

Finally

$$M_c = M_i\left\{1+\frac{\mu}{2}\left[\frac{\cos 2\beta-\sigma^2}{\sigma^2}+2\frac{1-\sigma^2\cos 2\beta}{1-\sigma^4}-\frac{1-\sigma^2\cos 2\beta}{\sigma^4} + \frac{1-2\sigma^2\cos 2\beta+\sigma^4}{2\sigma^6}\log\frac{1+\sigma^2}{1-\sigma^2}\right]-\frac{\mu}{2}\frac{\cosh 2\alpha-\cos 2\beta}{\sinh 2\alpha}\right\}$$

where

$$M_i = \frac{\pi}{2}\rho_0 v_0^2 c^2 \sin 2\beta$$

Replacing $e^{-2\alpha}$ by σ^2:

$$M_c = M_i\left[1+\frac{\mu}{2} + \frac{\mu}{2}(1-2\sigma^2\cos 2\beta+\sigma^4)\left(\frac{1}{2\sigma^6}\log\frac{1+\sigma^2}{1-\sigma^2}-\frac{1}{\sigma^4}\right)\right]$$

This expression agrees with equation (21) when the circulation is taken to be zero.

APPENDIX B

THE CALCULATION OF $\Delta v_s/v_0$ FOR AN ELLIPTIC CYLINDER AT $\delta=\pi$

Equation (13) of reference 2 gives an expression for $\Delta v_s/v_0$ independent of the shape of the obstacle. In order to evaluate $\Delta v_s/v_0$, it is necessary to know the Fourier development of v_i^2/v_0^2. For an elliptic cylinder, with the circulation chosen so that the rear stagnation point occurs at the end of the major axis, it may be easily shown by means of the Joukowski transformation and the complex velocity potential of the flow about a circular cylinder that:

$$\frac{v_i^2}{v_0^2} = \frac{(1+2\lambda\cos\theta+\lambda^2)[1-2\lambda\cos(\theta+2\beta)+\lambda^2]}{1-2\sigma^2\lambda^2\cos 2\theta+\sigma^4\lambda^4}$$

$$=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)$$

where

$\lambda=r_0/r$, $\sigma=c/2r_0$, β is the angle of attack, and r, θ are the polar coordinates of a point in the region of flow.

The Fourier series for v_i^2/v_0^2 is obtained by making use of the expansion

$$\frac{1}{1-2\sigma^2\lambda^2\cos 2\theta+\sigma^4\lambda^4}=\frac{1}{1-\sigma^4\lambda^4}\left[1+2\sum_{n=1}^{\infty}(\sigma^2\lambda^2)^n\cos 2n\theta\right]$$

and it may be shown without difficulty that

$$a_0=\frac{2(1+\lambda^2)^2}{1-\sigma^4\lambda^4}-\frac{4\lambda^2\cos 2\beta}{1-\sigma^2\lambda^2}$$

$$a_{2n}=2\left[\frac{(1+\lambda^2)^2}{1-\sigma^4\lambda^4}-\frac{1+\sigma^2\lambda^2}{\sigma^2(1-\sigma^2\lambda^2)}\cos 2\beta\right](\sigma\lambda)^{2n}$$

$$a_{2n+1}=\frac{4\lambda(1+\lambda^2)\sin^2\beta}{1-\sigma^2\lambda^2}(\sigma\lambda)^{2n}$$

$$b_{2n}=2\lambda^2(\sigma\lambda)^{2n-2}\sin 2\beta$$

$$b_{2n+1}=\frac{2\lambda(1+\lambda^2)\sin 2\beta}{1+\sigma^2\lambda^2}(\sigma\lambda)^{2n}$$

Equation (13) of reference 2 may then be written as

$$\left(\frac{\Delta v_s}{v_0}\right)_{\delta=\pi}=\frac{\mu}{2}\left[\sin\beta+\cos\beta\sum_{n=1}^{\infty}(2n+1)\int_0^1\lambda^{2n+1}b_{2n}d\lambda\right.$$
$$-2\cos\beta\sum_{n=0}^{\infty}(n+1)\int_0^1\lambda^{2(n+1)}b_{2n+1}d\lambda$$
$$-\cos\beta\sum_{n=0}^{\infty}(2n+1)\int_0^1\lambda^{2n-1}b_{2n+2}d\lambda$$
$$+2\cos\beta\sum_{n=0}^{\infty}(n+1)\int_0^1\lambda^{2n}b_{2n+3}d\lambda-\sin\beta\int_0^1\lambda a_0 d\lambda$$
$$-\sin\beta\sum_{n=1}^{\infty}(2n+1)\int_0^1\lambda^{2n+1}a_{2n}d\lambda$$
$$+2\sin\beta\sum_{n=0}^{\infty}(n+1)\int_0^1\lambda^{2(n+1)}a_{2n+1}d\lambda$$
$$-\sin\beta\sum_{n=0}^{\infty}(2n+1)\int_0^1\lambda^{2n-1}a_{2n+2}d\lambda$$
$$+2\sin\beta\sum_{n=0}^{\infty}(n+1)\int_0^1\lambda^{2n}a_{2n+3}d\lambda$$
$$+4\sin\beta\sum_{n=0}^{\infty}(n+1)\int_0^1\lambda^{2n+1}a_{2n+2}d\lambda$$
$$\left.-2\sin\beta\sum_{n=0}^{\infty}(2n+1)\int_0^1\lambda^{2n}a_{2n+1}d\lambda\right]$$

If the expressions for a_n and b_n are inserted and the integrations performed, it follows after considerable but straightforward labor that

$$\left(\frac{\Delta v_s}{v_0}\right)_{\delta=\pi}=\frac{\mu}{2}\left\{\sin\beta-\frac{1}{\sigma^2}\sin 3\beta+\frac{1}{\sigma^3}[(1+\sigma)^2\log(1+\sigma)\right.$$
$$-(1-\sigma)^2\log(1-\sigma)](\sin\beta+\sin 3\beta)$$
$$-\frac{1}{\sigma^4}[(1+\sigma^2)^2\log(1+\sigma^2)-(1-\sigma^2)^2\log(1-\sigma^2)]\sin\beta$$
$$\left.+\frac{8\log(1-\sigma^4)}{\sigma^2}\sin^3\beta-\frac{12}{\sigma^2}\sin\beta\log(1+\sigma^2)\right\}$$

The additional circulation ΔK is given by

$$\Delta K = -\left(\frac{\Delta v_\delta}{v_0}\right)_{\delta=\pi}$$

(See section on the elliptic profile.)
Therefore, with $K_i = 2 \sin \beta$, it follows that

$$\frac{\Delta \Gamma}{\Gamma_i} = \frac{\Delta K}{K_i} = \frac{\mu}{2}(M + N \sin^2\beta)$$

where

$$M = \frac{3-\sigma^2}{2\sigma^2} + \frac{2}{\sigma^2} \log(1+\sigma^2) + \frac{1+6\sigma^2+\sigma^4}{4\sigma^4} \log \frac{1+\sigma^2}{1-\sigma^2} - \frac{1+\sigma^2}{\sigma^3} \log \frac{1+\sigma}{1-\sigma}$$

$$N = \frac{1+\sigma^2}{\sigma^3} \log \frac{1+\sigma}{1-\sigma} - \frac{2}{\sigma^2}[1 + \log(1+\sigma^2)]$$

It is interesting to note that, for the limiting case of a circular cylinder $\sigma=0$, the foregoing equation yields

$$\left(\frac{\Delta\Gamma}{\Gamma_i}\right)_{circle} = \frac{\mu}{2}\left(\frac{11}{6} + \frac{2}{3}\sin^2\beta\right)$$

and compares with Poggi's result (reference 5)

$$\left(\frac{\Delta\Gamma}{\Gamma_i}\right)_{circle} = \frac{11}{12}\mu$$

For the straight-line profile $\sigma=1$, it is seen that $M=1$ and $N=\infty$. For an infinitely thin profile and a vanishingly small angle of attack, however,

$$\left(\frac{\Delta\Gamma}{\Gamma_i}\right)_{line} = \frac{1}{2}\mu$$

and compares with Glauert's formula (reference 3):

$$\frac{\Delta\Gamma}{\Gamma_i} = -1 + \frac{1}{\sqrt{1-\mu}} = \frac{1}{2}\mu + \ldots$$

APPENDIX C

EVALUATION OF I

$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{\overline{A}_n A_{n+1}}{r_0^{2n+1}} = \sum_{n=1}^{\infty}\frac{\overline{A}_n A_{n+1}}{n r_0^{2n+1}} + \sum_{n=2}^{\infty}\frac{\overline{A}_{n-1} A_n}{n r_0^{2n-1}}$$
$$= \frac{3}{2}\frac{\overline{A}_1 A_2}{r_0^3} + \frac{5}{6}\frac{\overline{A}_2 A_3}{r_0^5} + \frac{a^4}{r_0^4 k^5}(1-2k\cos 2\beta + k^2)\sum_{n=3}^{\infty}\frac{k^{2n}}{n}$$
$$+ \frac{a^4}{r_0^4 k^7}(1-2k\cos 2\beta+k^2)\sum_{n=4}^{\infty}\frac{k^{2n}}{n}$$

Since

$-\sum_{n=1}^{\infty}\frac{k^{2n}}{n} = \log(1-k^2)$, $A_1 = 2ir_0 \sin\beta$, $A_2 = a^2 e^{i\beta} - r_0^2 e^{-i\beta}$
and $A_3 = a^2 r_0 (k e^{i\beta} - e^{-i\beta})$, it follows that

$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{\overline{A}_n A_{n+1}}{r_0^{2n+1}}$$
$$= -\frac{a^4}{r_0^4}\frac{1+k^2}{k^7}(1-2k\cos 2\beta+k^2)\log(1-k^2)$$
$$-\frac{a^4}{r_0^4 k^5}(1-2k\cos 2\beta+k^2)\left(1+\frac{3}{2}k^2+\frac{5}{6}k^4\right)$$
$$-3i\sin\beta\left(\frac{a^2}{r_0^2}e^{i\beta}-e^{-i\beta}\right)$$
$$+\frac{5}{6}\frac{a^2}{r_0^2}\left(\frac{a^2}{r_0^2}e^{-i\beta}-e^{i\beta}\right)(ke^{i\beta}-e^{-i\beta})$$

Therefore

$$I = \text{R. P. } i\frac{1-e^{2i\beta}}{2\sin 2\beta}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{\overline{A}_n A_{n+1}}{r_0^{2n+1}}$$
$$= -\frac{a^4}{r_0^4}\frac{1+k^2}{2k^7}(1-2k\cos 2\beta+k^2)\log(1-k^2)$$
$$-\frac{a^4}{2r_0^4 k^5}(1-2k\cos 2\beta+k^2)\left(1+\frac{3}{2}k^2+\frac{5}{6}k^4\right)$$
$$+\frac{a^2}{6r_0^2}(9+5k)(1-\cos 2\beta)+\frac{5}{12}\frac{a^2}{r_0^2}(1-k)\left(1-\frac{a^2}{r_0^2}\right)$$

Now

$$\frac{a}{r_0} = \frac{1}{1+\epsilon} \text{ and } k = \frac{1-\epsilon}{1+\epsilon}$$

Therefore

$$\epsilon = \frac{1-k}{1+k} \text{ and } \frac{a}{r_0} = \frac{1+k}{2}$$

Hence

$$(1+\epsilon)^2 I = \frac{(1+k)^2}{8k^5}(1-2k\cos 2\beta+k^2)\left[\frac{1+k^2}{k^2}\log\frac{1}{1-k^2} - \left(1+\frac{3}{2}k^2+\frac{5}{6}k^4\right)\right] + \frac{1}{6}(9+5k)(1-\cos 2\beta)$$
$$+\frac{5}{48}(3+k)(1-k)^2$$

For small values of k, or thick airfoils,

$$(1+\epsilon)^2 I = (1+k)^2(1-2k\cos 2\beta+k^2)\left(\frac{7}{96}k+\frac{9}{160}k^3 + \frac{11}{240}k^5 + \ldots\right)+\frac{1}{6}(9+5k)(1-\cos 2\beta)$$
$$+\frac{5}{48}(3+k)(1-k)^2$$

EVALUATION OF J

$$\sum_{n=1}^{\infty}\frac{\overline{A}_n A_{n+2}}{(n+1)r_0^{2n+2}} = \frac{\overline{A}_1 A_3}{2r_0^4} + \frac{\overline{A}_3 A_4}{3r_0^6} + \sum_{n=4}^{\infty}\frac{\overline{A}_{n-1}A_{n+1}}{nr_0^{2n}}$$
$$= \frac{a^4(1-2k\cos 2\beta+k^2)}{r_0^4 k^6}\sum_{n=4}^{\infty}\frac{k^{2n}}{n} - i\frac{a^2}{r_0^2}\sin\beta(ke^{i\beta}-e^{-i\beta})$$
$$+\frac{1}{3}\frac{a^2 k}{r_0^2}(ke^{i\beta}-e^{-i\beta})\left(\frac{a^2}{r_0^2}e^{-i\beta}-e^{i\beta}\right)$$
$$= -\frac{a^4(1-2k\cos 2\beta+k^2)}{r_0^4 k^6}\log(1-k^2)$$
$$-\frac{a^4(1-2k\cos 2\beta+k^2)}{r_0^4 k^4}\left(1+\frac{k^2}{2}+\frac{k^4}{3}\right) - i\frac{a^2}{r_0^2}\sin\beta(ke^{i\beta}-e^{-i\beta})$$
$$+\frac{1}{3}\frac{a^2 k}{r_0^2}(ke^{i\beta}-e^{-i\beta})\left(\frac{a^2}{r_0^2}e^{-i\beta}-e^{i\beta}\right)$$

Therefore

$$J=-\text{R. P.}\frac{ie^{2i\beta}}{\sin 2\beta}\sum_{n=1}^{\infty}\frac{\bar{A}_n A_{n+2}}{(n+1)r_0^{2n+2}}$$

$$=-\frac{a^4(1-2k\cos 2\beta+k^2)}{r_0^4k^6}\log(1-k^2)$$

$$-\frac{a^4(1-2k\cos 2\beta+k^2)}{r_0^4k^4}\left(1+\frac{k^2}{2}+\frac{k^4}{3}\right)$$

$$+\frac{a^2}{6r_0^2}(3+2k)(1-2k\cos 2\beta+k^2)$$

$$+\frac{a^2k}{2r_0^2}(1-k)+\frac{1}{3}\frac{a^2k^2}{r_0^2}\left(\frac{a^2}{r_0^2}-k\right)$$

and

$$(1+\epsilon)^2J=\frac{(1+k)^2}{4k^4}(1-2k\cos 2\beta+k^2)\left[\frac{1}{k^2}\log\frac{1}{1-k^2}\right.$$
$$\left.-\left(1+\frac{k^2}{2}+\frac{k^4}{3}\right)\right]+\frac{1}{6}(3+2k)(1-2k\cos 2\beta+k^2)$$
$$+\frac{1}{2}k(1-k)+\frac{1}{12}k^2(1-k)^2$$

For small values of k, or thick airfoils,

$$(1+\epsilon)^2J=(1+k)^2(1-2k\cos 2\beta+k^2)\left(\frac{1}{16}k^2\right.$$
$$\left.+\frac{1}{20}k^4+\frac{1}{24}k^6+\ \ldots\right)+\frac{1}{6}(3+2k)(1-2k\cos 2\beta+k^2)$$
$$+\frac{1}{2}k(1-k)+\frac{1}{12}k^2(1-k)^2$$

APPENDIX D

EVALUATION OF $(\Delta v_s/v_0)_{s=\pi}$ FOR THE CASE OF A JOUKOWSKI PROFILE

According to equation (22) of reference 2, $(\Delta v_s/v_0)_{s=\pi}$ takes the following form:

$$\left(\frac{\Delta v_s}{v_0}\right)_{s=\pi}=\frac{\mu}{2}\sin\beta\left\{-\frac{1}{2}+\frac{h}{k^2}\left(h+2k+2kh^2-\frac{5}{6}h^3\right)\right.$$
$$+\frac{(h+k)^2}{k^3}\left(-3-3kh+\frac{3}{2}k-h^2k^2+\frac{5}{6}kh^2\right)$$
$$+\frac{2h^2(1-k)^2}{k^5}\left[1-\frac{1}{2}k^2+\frac{1-k^2}{k^2}\log(1-k^2)\right]$$
$$+\frac{8h(1-h)^3(2-3h)}{k^5}\left[kh^2-\frac{1-h}{2}\right.$$
$$\left.+(1-h)^2+2h(1-h)\log(1+k)\right]$$
$$\left.+\frac{4h^2(1-h)^4}{k^6}[\log(1-k)+4h^2\log(1+k)-k^2]\right\}$$
$$+\frac{\mu}{2}\sin^3\beta\left\{\frac{4(h+k)^2}{k^3}\left(1+hk+\frac{1}{3}h^2k^2\right)\right.$$
$$+\frac{4h^2}{k^2}\left[-1+\frac{k}{2}-\frac{1}{3}hk+\frac{2}{k^2}+\frac{2(1-k^2)}{k^4}\log(1-k^2)\right]$$
$$+\frac{8(1-h)^3}{k^3}\left[kh^2-\frac{1-h}{2}+(1-h)^2+2h(1-h)\log(1+k)\right]$$
$$\left.+\frac{4(1-h)^4}{k^5}[\log(1-k)+4h^2\log(1+k)-k^2]\right\}$$

where

$$h=\frac{\epsilon}{1+\epsilon}\text{ and }k=\frac{1-\epsilon}{1+\epsilon}$$

When this expression for $(\Delta v_s/v_0)_{s=\pi}$ was obtained, a slight error was found in equations (19) of reference 2. The expressions for $a_1-\bar{a}_1$ and $b_1-\bar{b}_1$ should be as follows:

$$a_1-\bar{a}_1=\frac{2\lambda\cos 2\beta}{k^2}\left[h(h+2k)+2h^3k\lambda^2\right]$$
$$-\frac{2h^2\lambda}{k}\frac{1+\lambda^2(1-2k\cos 2\beta)}{1-k^2\lambda^2}$$

and

$$b_1-\bar{b}_1=-\frac{2\lambda\sin 2\beta}{k^2}\left[h(h+2k)+2kh^3\lambda^2\right]$$

It is to be noted in the expression for $(\Delta v_s/v_0)_{s=\pi}$ that most of the terms contain powers of k in the denominator. It appears at first, then, that the coefficients of $\sin\beta$ and $\sin^3\beta$ may become infinite for $k=0$. This apparent difficulty disappears, however, when $(\Delta v_s/v_0)_{s=\pi}$ is expressed as a power series in k. It is then found that the terms involving reciprocal powers of k cancel and the following expression results:

$$\left(\frac{\Delta v_s}{v_0}\right)_{s=\pi}$$
$$=\frac{\mu}{2}\sin\beta\left(-\frac{689}{192}+\frac{309}{160}k-\frac{289}{672}k^2+\frac{7}{32}k^3-\frac{191}{640}k^4+\ \ldots\right)$$
$$+\frac{\mu}{2}\sin^3\beta\left(-\frac{41}{24}-\frac{689}{240}k-\frac{13}{24}k^2-\frac{1373}{1680}k^3-\frac{163}{560}k^4-\ \ldots\right)$$

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